A Hemivariational Inequality Approach to the Navier-Stokes-Darcy Model

Stanislaw Migórski

Jagiellonian University in Krakow, Poland
Faculty of Mathematics and Computer Science

stanislaw.migorski@uj.edu.pl

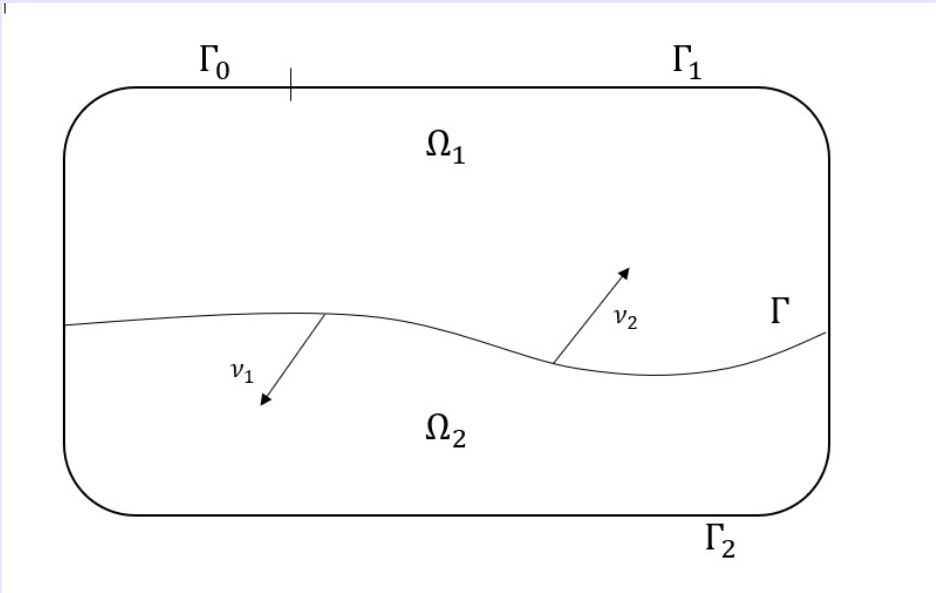
Motivation

- Filtration of blood through arterial vessel walls
- Groundwater flow and pollutant transport through porous soils
- Movement of oil, gas and water in rock formations
- Percolation of water of hydrological basin through rocks and sand

Areas which profit from the model

- Biomedicine
- Coastal engineering
- Environmental and chemical engineering
- Geophysics
- Hydrology
- Petroleum industry

A sketch of a general geometry

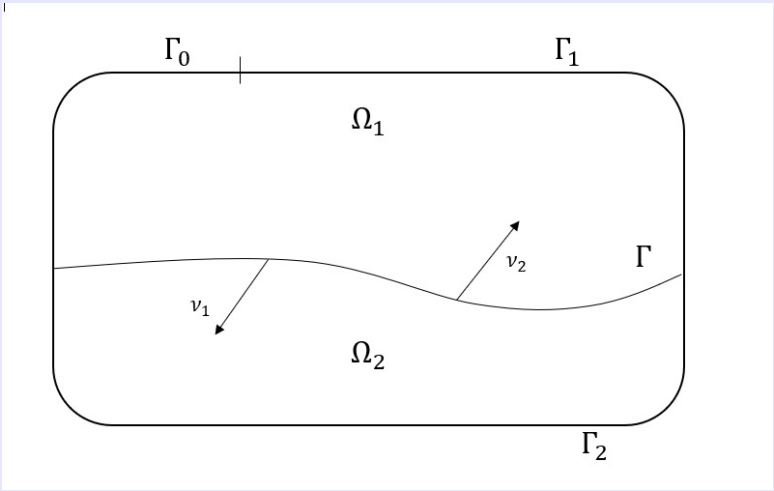


Free-flow region Ω_1: an incompressible viscous fluid steady flow is modeled by the Navier-Stokes equations

Porous media region Ω_2: the flow is described by Darcy's law

The permeable interface Γ: regions Ω_1 and Ω_2 are separated by Γ, where two kinds of fluid flow interact each other.

A geometry of the model



Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $d = 2, 3$. We suppose Ω is partitioned into two nonempty subdomains Ω_1 and Ω_2 which both boundaries are also Lipschitz continuous. Let $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma \neq \emptyset$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Moreover, we define

$$\partial\Omega_1 = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}, \quad \Gamma_0 \cap \Gamma_1 = \emptyset, \quad (\Gamma_0 \cup \Gamma_1) \cap \Gamma = \emptyset,$$

and

$$\partial\Omega_2 = \overline{\Gamma}_2 \cup \overline{\Gamma}, \quad \Gamma_2 \cap \Gamma = \emptyset,$$

where the measures of Γ_0 and Γ_2 are nonzero. We denote by $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ the unit outward normal vectors on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Clearly $\boldsymbol{\nu}_1 = -\boldsymbol{\nu}_2$, and so we write $\boldsymbol{\nu} = \boldsymbol{\nu}_1$.

Free flow in the region Ω_1

The free flow in the region Ω_1 is governed by the Navier-Stokes equations of the form

$$-\operatorname{Div}\mathbb{S}(\mathbb{D}\boldsymbol{u}_1)+\operatorname{Div}(\boldsymbol{u}_1\otimes\boldsymbol{u}_1)+\nabla p_1=\boldsymbol{f}\ \text{ in }\quad \Omega_1, \tag{1}$$

$$\operatorname{div}\boldsymbol{u}_1=0\qquad\qquad\qquad\qquad\qquad\qquad\ \text{ in }\quad \Omega_1, \tag{2}$$

with the following boundary conditions

$$\boldsymbol{u}_1=0\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \text{ on }\quad \Gamma_0, \tag{3}$$

$$u_{1\nu}=0,\quad -\boldsymbol{\tau}_\tau(\boldsymbol{u}_1)\in k(\boldsymbol{u}_{1\tau})\,\partial j_\tau(\boldsymbol{u}_{1\tau})\quad \text{ on }\quad \Gamma_1. \tag{4}$$

Here, the unknows are the Stokes velocity $\boldsymbol{u}_1\colon\Omega_1\to\mathbb{R}^d$, the (kinematic) pressure $p_1\colon\Omega_1\to\mathbb{R}$, and the extra stress tensor $\mathbb{S}\colon\mathbb{M}^d\to\mathbb{M}^d$. We denote by $\mathbb{M}^d$ the class of symmetric $d\times d$ matrices,

$$\mathbb{D}\boldsymbol{u}_1=\frac{1}{2}(\nabla\boldsymbol{u}_1+(\nabla\boldsymbol{u}_1)^\top) \tag{5}$$

represents the symmetric part of the velocity gradient (the deformation rate tensor).

Free flow in the region Ω_1

Notation

The total stress tensor is defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}_1, p_1) = -p_1 \mathbb{I} + \mathbb{S}(\mathbb{D}\boldsymbol{u}_1) \quad \text{in} \quad \Omega_1,$$

where $\mathbb{I}$ is the $d \times d$ identity matrix. Furthermore, we introduce the traction vector given by

$$\boldsymbol{\tau}(\boldsymbol{u}_1, p_1) = \boldsymbol{\sigma}(\boldsymbol{u}_1, p_1)\,\boldsymbol{\nu} \quad \text{on the boundary},$$

and $\boldsymbol{f} : \Omega_1 \to \mathbb{R}^d$ is the volume density of given forces.
The normal and tangential components of the velocity and of the traction on the boundary are denoted by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, and by $\tau_\nu(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}, p) \cdot \boldsymbol{\nu}$, $\boldsymbol{\tau}_\tau(\boldsymbol{u}) = \boldsymbol{\tau}(\boldsymbol{u}, p) - \tau_\nu(\boldsymbol{u}, p)\boldsymbol{\nu}$, respectively. We have

$$\mathbb{S}_\nu(\boldsymbol{u}) = \tau_\nu(\boldsymbol{u}, p) + p, \quad \mathbb{S}_\tau(\boldsymbol{u}) = \boldsymbol{\tau}_\tau(\boldsymbol{u}) \quad \text{on the boundary.} \tag{6}$$

Then $\boldsymbol{\tau}$ and τ_ν do depend on the pressure p, and $\boldsymbol{\tau}_\tau$ is independent of p.

Free flow in the region Ω_1

The boundary conditions

$$\boldsymbol{u}_1 = 0 \quad \text{on} \quad \Gamma_0,$$

and

$$u_{1\nu} = 0 \quad \text{on} \quad \Gamma_1,$$

are chosen to be homogeneous only for simplicity of the presentation and the other type of boundary conditions could have been chosen as well.

The condition $u_{1\nu} = 0$ on Γ_1 is called the no leak (impermeability) boundary condition and means that the fluid can not pass through Γ_1 outside the domain Ω_1.

The condition

$$-\boldsymbol{\tau}_\tau(\boldsymbol{u}_1) \in k(\boldsymbol{u}_{1\tau})\, \partial j_\tau(\boldsymbol{u}_{1\tau}) \quad \text{on} \quad \Gamma_1.$$

is a "friction type" boundary condition between the tangential traction force $\boldsymbol{\tau}_\tau(\boldsymbol{u}_1)$ and the tangential velocity $\boldsymbol{u}_{1\tau}$, also known as a slip boundary condition.

Porous medium region Ω_2

The viscous fluid flow in Ω_2 is described by the linearized Darcy model

$$\operatorname{div} \boldsymbol{u}_2 = g \quad \text{in} \quad \Omega_2, \tag{7}$$

$$\boldsymbol{u}_2 = -K\nabla\varphi \quad \text{in} \quad \Omega_2, \tag{8}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_2. \tag{9}$$

It includes the saturated flow in (7) and Darcy's law in (8). The unknows are the Darcy velocity (the fluid discharge rate in the porous media) $\boldsymbol{u}_2 \colon \Omega_2 \to \mathbb{R}^d$ and the Darcy pressure (hydraulic head) $\varphi \colon \Omega_2 \to \mathbb{R}$. The prescribed data are the external force density $g \colon \Omega_2 \to \mathbb{R}$ and a bounded and positive definite matrix $K \colon \Omega_2 \to \mathbb{M}^d$ corresponding to the permeability of the region Ω_2.

From (7) and (8), we get the Darcy equation $-\operatorname{div}(K\nabla\varphi) = g$ in Ω_2. The Dirichlet boundary condition (9) is prescribed for simplicity. Other boundary conditions, e.g., the nonhomogeneous Neumann condition on a part of Γ_2, can be considered as well.

Interface transmission conditions

A key part of the model are the interface coupling conditions. In addition to the boundary conditions on the outer boundary of $\partial\Omega$ stated in (3), (4), and (9), we complement the problem with the following interface transmission conditions on Γ which couple the Navier-Stokes flow and the Darcy flow.

$$\begin{cases} \text{(a)} & \boldsymbol{u}_1 \cdot \boldsymbol{\nu} = \boldsymbol{u}_2 \cdot \boldsymbol{\nu}, \\ \text{(b)} & -\boldsymbol{\tau}_\nu(\boldsymbol{u}_1, p_1) + \frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = \varphi, \qquad \text{on} \quad \Gamma. \\ \text{(c)} & \boldsymbol{u}_{1\tau} = -\alpha\, \boldsymbol{\tau}_\tau(\boldsymbol{u}_1). \end{cases} \tag{10}$$

A parameter $\alpha > 0$ in (10) (c) is usually obtained from experimental data.

* Condition (10)(c) can be generalized to

$$-\tau_\tau(\boldsymbol{u}_1) \in m(\boldsymbol{u}_{1\tau})\, \partial g_\tau(\boldsymbol{u}_{1\tau}) \quad \text{on} \quad \Gamma,$$

where $g_\tau(\boldsymbol{x}, \cdot)\colon \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz.

Interface transmission conditions

The condition (10) (a)

$$\boldsymbol{u}_1 \cdot \boldsymbol{\nu} = \boldsymbol{u}_2 \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma$$

assigned at the permeable interface Γ guarantees the continuity of the normal velocity. It means that

$$\boldsymbol{u}_1 \cdot \boldsymbol{\nu} = -K \nabla \varphi \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma,$$

i.e., the flow is balanced on both sides of the transmission surface, so that the velocities across Γ are equivalent. Therefore, (10) (a) represents the conservation of mass and signifies the continuity in normal velocity on the interface.

Interface transmission conditions

The condition (10) (b) is known as the Lions interface condition

$$-\boldsymbol{\tau}_\nu(\boldsymbol{u}_1, p_1)+\frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = \varphi \quad \text{on} \quad \Gamma$$

and describes the balance of normal forces across the interface Γ.

- In the literature, there are two different ways to write this balance: with inertial forces, as in (10) (b), or without inertial forces in which the term $\frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1)$ representing inertial forces is neglected.
- Condition (10) (b) without inertial forces has been treated in the linear Stokes-Darcy model as being more physically correct but less tractable analytically.
- The incorporation of the inertial term in the interface condition (10) (b) is essential in conducting the nonlinear stability analysis in numerical computations.

Interface transmission conditions (3)

The condition (10) (c) has originated from the Navier slip boundary condition introduced in 1823, and has emerged from the celebrated Beavers-Joseph condition 1967. The latter agrees best with experimental data and states that the slip velocity along Γ is proportional to the shear stress, i.e.,

$$(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \tau \approx \boldsymbol{\tau}(\boldsymbol{u}_1, p_1) \cdot \tau \quad \text{on} \quad \Gamma.$$

The paper of Saffman (1971), by a statistical approach, supports the claim that the term $\boldsymbol{u}_2$ is often negligible with respect to $\boldsymbol{u}_1$. Without this term, we arrive to the Beavers-Joseph-Saffman (BJS) law, and is given by (10) (c).

In conclusion, we obtain the classical formulation of the NSD problem.

Problem

Find $(\boldsymbol{u}_1, p_1, \mathbb{S})$ *and* $(\boldsymbol{u}_2, \varphi)$ *such that* (1)–(4), (7)–(9) *and* (10) *are satisfied.*

Surface roughness and boundary conditions

- H. Wu, W. Chen, Z. Jiang, Slip boundary conditions for rough surfaces, *Chinese J. Aeronautics* **36** (2023), 239–249.
- G.A. Zampogna, J. Magnaudet, A. Bottaro, Generalized slip condition over rough surfaces, *J. Fluid Mechanics* **858** (2019), 407–436.

"It is a challenging work since there is still a lack of an appropriate theoretical description of the realistic surface roughness, and thus it is difficult to clarify the role of the roughness". "What, however, still remains a subject of hot debates is that how to correct the flow near the interface, and what would be the amplitude of these corrections".

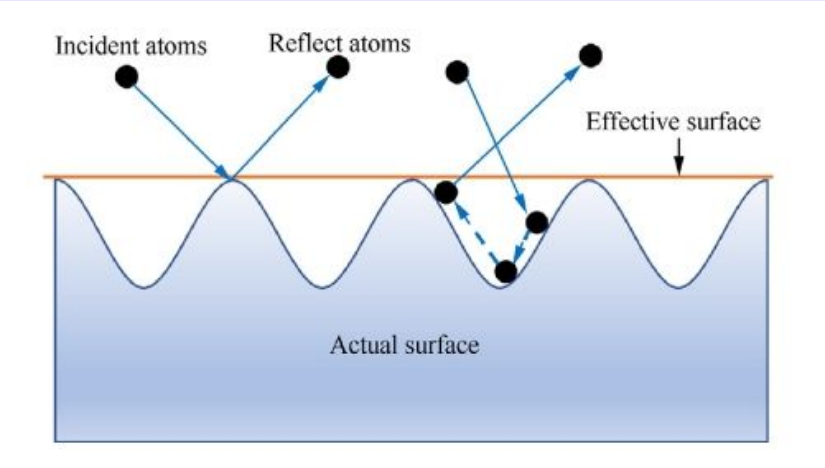


Figure: Reflective behavior of molecules on rough surfaces

Variational formulation

Let the functions in the coupled problem (1)–(4), (7)–(9) and (10) be sufficiently smooth and satisfy the classical formulation.

For the Navier-Stokes velocity and the Darcy pressure, we introduce the following spaces

$$V_S = \{\boldsymbol{v} \in H^1(\Omega_1; \mathbb{R}^d) \mid \boldsymbol{v} = 0 \text{ on } \Gamma_0,\ v_\nu = 0 \text{ on } \Gamma_1,\ \mathrm{div}\boldsymbol{v} = 0 \text{ in } \Omega_1\},$$

$$P_D = \{\psi \in H^1(\Omega_2) \mid \psi = 0 \text{ on } \Gamma_2\},$$

and $L^2_0(\Omega_1) = \{q \in L^2(\Omega_1) \mid \int_{\Omega_1} q\,dx = 0\}$. Let $V := V_S \times P_D$ be a reflexive Hilbert space endowed with the inner product

$$\langle(\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)\rangle = \int_{\Omega_1} \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v})\,dx + \int_{\Omega_2} \nabla\varphi \cdot \nabla\psi\,dx$$

for $(\boldsymbol{u}, \varphi)$, $(\boldsymbol{v}, \psi) \in V$.

Variational formulation

Let $b\colon H^1(\Omega_1;\mathbb{R}^d)^3 \to \mathbb{R}$ be the trilinear form defined by

$$b(\boldsymbol{v};\boldsymbol{u},\boldsymbol{z}) = \sum_{i,j=1}^{d} \int_{\Omega_1} v_j \frac{\partial u_i}{\partial x_j} z_i \, dx \quad \text{for} \quad \boldsymbol{v},\boldsymbol{u},\boldsymbol{z} \in H^1(\Omega_1;\mathbb{R}^d)$$

corresponding to the convective term in the Navier-Stokes equation.

Variational formulation

The following Navier-Stokes-Darcy (NSD) hemivariational inequality of elliptic type is obtained. Note that the Navier-Stokes pressure p_1 and the Darcy velocity $\boldsymbol{u}_2$ have been eliminated.

Problem

Find the velocity $\boldsymbol{u}_1 : \Omega_1 \to \mathbb{R}^d$ *and the pressure* $\varphi : \Omega_2 \to \mathbb{R}$ *such that*

$$\int_{\Omega_1} \mathbb{S}(\mathbb{D}\boldsymbol{u}_1) : \mathbb{D}\boldsymbol{v}\, dx + b(\boldsymbol{u}_1; \boldsymbol{u}_1, \boldsymbol{v}) + \int_{\Omega_2} K\nabla\varphi \cdot \nabla\psi\, dx$$
$$+ \int_{\Gamma_1} k(\boldsymbol{u}_{1\tau})\, j_\tau^0(\boldsymbol{u}_{1\tau}; \boldsymbol{v}_\tau)\, d\Gamma + \int_\Gamma \left(\varphi - \frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1)\right) v_\nu\, d\Gamma$$
$$+ \frac{1}{\alpha} \int_\Gamma (\boldsymbol{u}_1 \cdot \tau) \cdot \boldsymbol{v}_\tau\, d\Gamma - \int_\Gamma u_{1\nu}\, \psi\, d\Gamma \geq \int_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\Omega_2} g\, \psi\, dx$$

for all $(\boldsymbol{v}, \psi) \in V = V_S \times P_D$.

Solvability of the NSD hemivariational inequality

H($\mathbb{S}$): The function $\mathbb{S}\colon \Omega_1 \times \mathbb{M}^d \to \mathbb{M}^d$ is such that

(i) $\mathbb{S}(\cdot, \mathbb{D})$ is measurable on Ω_1 for all $\mathbb{D} \in \mathbb{M}^d$,

(ii) $\mathbb{S}(\boldsymbol{x}, \cdot)$ is continuous and monotone on $\mathbb{M}^d$ for a.e. $\boldsymbol{x} \in \Omega_1$,

(iii) $\|\mathbb{S}(\boldsymbol{x}, \mathbb{D})\| \leq a_1(\boldsymbol{x}) + a_2 \|\mathbb{D}\|$ for all $\mathbb{D} \in \mathbb{M}^d$, a.e. $\boldsymbol{x} \in \Omega_1$ with $a_1 \in L^2(\Omega_1)$, $a_1,\ a_2 > 0$,

(iv) $\mathbb{S}(\boldsymbol{x}, \mathbb{D}) : \mathbb{D} \geq \alpha_S \|\mathbb{D}\|^2$ for all $\mathbb{D} \in \mathbb{M}^d$, a.e. $\boldsymbol{x} \in \Omega_1$ with $\alpha_S > 0$.

H($\mathbf{f}, \mathbf{g}$): Let $\boldsymbol{f} \in L^2(\Omega_1; \mathbb{R}^d)$, $g \in L^2(\Omega_2)$.

H($\mathbf{j}_\tau$): The function $j_\tau\colon \Gamma_1 \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j_\tau(\cdot, \boldsymbol{\xi})$ is measurable on Γ_1 for all $\boldsymbol{\xi} \in \mathbb{R}^d$,

(ii) $j_\tau(\boldsymbol{x}, \cdot)$ is locally Lipschitz for a.e. $\boldsymbol{x} \in \Gamma_1$,

(iii) $\|\partial j_\tau(\boldsymbol{x}, \boldsymbol{\xi})\| \leq b_1(\boldsymbol{x}) + b_2 \|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_1$ with $b_1 \in L^2(\Gamma_1)$, $b_1,\ b_2 \geq 0$.

Solvability of the NSD hemivariational inequality

H(k): The function $k\colon \Gamma_1 \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $k(\cdot, \boldsymbol{\xi})$ is measurable on Γ_1 for all $\boldsymbol{\xi} \in \mathbb{R}^d$,

(ii) $k(\boldsymbol{x}, \cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_1$,

(iii) $0 < k_0 \leq k(\boldsymbol{x}, \boldsymbol{\xi}) \leq k_1$ for $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_1$, where $k_0,\ k_1 \in \mathbb{R}$.

H(K): The permeability function $K\colon \Omega_2 \to \mathbb{M}^d$ is such that

(i) $K(\boldsymbol{x}) = \{k_{ij}(\boldsymbol{x})\}$, $k_{ij} \in L^\infty(\Omega_2)$ for $i,\ j = 1, \ldots, d$, a.e. $\boldsymbol{x} \in \Omega_2$,

(ii) $m_0 \|\boldsymbol{\xi}\|^2 \leq K(\boldsymbol{x})\,\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq m_1 \|\boldsymbol{\xi}\|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega_2$ with some $m_0,\ m_1 > 0$.

Theorem

Under $H(\mathbb{S}), H(f,g), H(j_\tau), H(k), H(K)$, and the smallness condition

$$\min\{\alpha_S, m_0\} > \sqrt{2}\, k_1\, b_2 \|M\|^2,$$

the NSD hemivariational inequality has a solution $(\boldsymbol{u}, \varphi) \in V$.

Formulation of the hemivariational inequality

Let $X_S = L^2(\Gamma_1; \mathbb{R}^d)$. We introduce the following operators and functionals defined by

$$A\colon V \to V^*, \quad \langle A(\boldsymbol{w}), \boldsymbol{z}\rangle = \int_{\Omega_1} \mathbb{S}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\, dx + \int_{\Omega_2} K\nabla\varphi \cdot \nabla\psi\, dx$$
$$\text{for } \boldsymbol{w} = (\boldsymbol{u}, \varphi),\ \boldsymbol{z} = (\boldsymbol{v}, \psi) \in V,$$

$$B[\cdot]\colon V_S \to V_S^*, \quad B[\boldsymbol{u}] = B(\boldsymbol{u}, \boldsymbol{u}),\ \langle B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{z}\rangle = b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{z})$$
$$\text{for } \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{z} \in V_S,$$

$$C\colon V \to V^*,$$
$$\langle C(\boldsymbol{w}), \boldsymbol{z}\rangle = \int_\Gamma \Big(\varphi - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u})\Big) v_\nu\, d\Gamma + \frac{1}{\alpha}\int_\Gamma \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau\, d\Gamma - \int_\Gamma u_\nu\, \psi\, d\Gamma$$
$$\text{for } \boldsymbol{w} = (\boldsymbol{u}, \varphi),\ \boldsymbol{z} = (\boldsymbol{v}, \psi) \in V,$$

$$T\colon V \to V^*, \quad \langle T(\boldsymbol{w}), \boldsymbol{z}\rangle = \langle A(\boldsymbol{w}), \boldsymbol{z}\rangle + \langle B(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{v}\rangle + \langle C(\boldsymbol{w}), \boldsymbol{z}\rangle$$
$$\text{for } \ \boldsymbol{w} = (\boldsymbol{u}, \varphi) \in V,\ \boldsymbol{z} = (\boldsymbol{v}, \psi) \in V,$$

Formulation of the hemivariational inequality

and

$$J\colon X_S \times X_S \to \mathbb{R}, \quad J(x_1, x_2) = \int_{\Gamma_1} k(x_1) j_\tau(x_2)\, d\Gamma \quad \text{for } x_1, x_2 \in X_S,$$

$$M\colon V_S \to X_S, \quad M\boldsymbol{v} = \boldsymbol{v}_\tau \quad \text{for } \boldsymbol{v} \in V_S,$$

$$G\colon V \to 2^{V^*}, \quad G(\boldsymbol{w}) = G(\boldsymbol{u}, \varphi) = M^* \partial J(M\boldsymbol{u}, M\boldsymbol{u})$$

$$\text{for } \boldsymbol{w} = (\boldsymbol{u}, \varphi) \in V.$$

Formulation of the hemivariational inequality

We consider the following operator inclusion. Find $\boldsymbol{w} = (\boldsymbol{u}, \varphi) \in V$ such that

$$T\boldsymbol{w} + G(\boldsymbol{w}) \ni F \quad \text{in} \quad V^*,$$

where $F \in V^*$ is defined by

$$\langle F(f, g), \boldsymbol{z} \rangle = \int_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v} \, dx + \int_{\Omega_2} g \, \psi \, dx \quad \text{for } \boldsymbol{z} = (\boldsymbol{v}, \psi) \in V.$$

We can prove that the multivalued operator $T + G \colon V \to 2^{V^*}$ is a bounded, pseudomonotone and coercive, and hence surjective.

Recovery of the Stokes pressure

We have a result saying that any sufficiently smooth solution to Problem 2 is a solution to Problem 1.
We need the following spaces

$$T_\tau := \{\boldsymbol{v}_\tau|_{\Gamma_1} \mid \boldsymbol{v} \in V_S\} \subsetneq H_\tau^{1/2}(\Gamma_1;\mathbb{R}^d),$$
$$T_\nu := \{v_\nu|_\Gamma \mid \boldsymbol{v} \in V_S\} \subsetneq H^{1/2}(\Gamma),$$

where

$$H_\tau^{1/2}(\Gamma_1;\mathbb{R}^d) = \{\boldsymbol{z} \in H^{1/2}(\Gamma_1;\mathbb{R}^d) \mid \boldsymbol{z}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_1\}$$

and

$$T_{\tau,\Gamma} := \{\boldsymbol{v}_\tau|_\Gamma \mid v \in V_S, \operatorname{supp}\boldsymbol{v}_\tau \subset \Gamma\},$$
$$T_{\tau,\Gamma_1} := \{\boldsymbol{v}_\tau|_{\Gamma_1} \mid v \in V_S, \operatorname{supp}\boldsymbol{v}_\tau \subset \Gamma_1\}.$$

The trace spaces T_τ and T_ν are closed linear spaces because V_S is a linear space and trace operators are continuous. Also, T_{τ,Γ_1} is a linear subspace of $H_\tau^{1/2}(\Gamma_1;\mathbb{R}^d)$. For each $\boldsymbol{v} \in V_S$, we denote its traces as $(\boldsymbol{v}_\tau, v_\nu) \in T_\tau \times T_\nu$.

Recovery of the Stokes pressure

Theorem

(1) *Let* $(\boldsymbol{u}_1, \varphi) \in V$ *be a solution to Problem* 2 *provided by Theorem* 3. *Then it fulfils the conditions of Problem* 1 *in the following sense: there is a unique* $p_1 \in L^2_0(\Omega_1)$ *such that*

$$-\operatorname{Div}\mathbb{S}(\mathbb{D}\boldsymbol{u}_1) + \operatorname{Div}(\boldsymbol{u}_1 \otimes \boldsymbol{u}_1) + \nabla p_1 = \boldsymbol{f} \quad \text{in} \quad \mathcal{D}'(\Omega_1) \tag{11}$$

$$-\operatorname{div}(K\nabla\varphi) = g \quad \text{in} \quad \mathcal{D}'(\Omega_2) \tag{12}$$

$$-\boldsymbol{\tau}_\nu(\boldsymbol{u}_1, p_1) + \frac{1}{2}\|\boldsymbol{u}_1\|^2 = \varphi \quad \text{on} \quad \Gamma \text{ in } T'_\nu \tag{13}$$

$$-\alpha\,\boldsymbol{\tau}_\tau(\boldsymbol{u}_1) = \boldsymbol{u}_{1\tau} \quad \text{on} \quad \Gamma \text{ in } T'_{\tau,\Gamma} \tag{14}$$

$$-\boldsymbol{\tau}_\tau(\boldsymbol{u}_1) \in k(\boldsymbol{u}_{1\tau})\,\partial j_\tau(\boldsymbol{u}_{1\tau}) \quad \text{on} \quad \Gamma_1 \text{ in } T'_{\tau,\Gamma_1} \tag{15}$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{\nu} = \boldsymbol{u}_2 \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma \text{ in } H^{-1/2}(\Gamma). \tag{16}$$

Recovery of the Stokes pressure

Theorem

(2) *If we assume an additional regularity*

$$\boldsymbol{u}_1 \in H^2(\Omega_1;\mathbb{R}^d), \quad \varphi \in H^2(\Omega_2), \quad \text{and} \quad p_1 \in H^1(\Omega_1), \tag{17}$$

then the partial differential equations (11) *and* (12) *in* Ω_1 *and* Ω_2, *respectively, hold a.e., and the interface and boundary conditions* (13)–(15) *on* Γ *and* Γ_1 *hold a.e.. If* (17) *holds and* $K \in W^{1,\infty}(\Omega_2;\mathbb{M}^d)$, *then the normal velocity continuity condition* $\boldsymbol{u}_1 \cdot \boldsymbol{\nu} = \boldsymbol{u}_2 \cdot \boldsymbol{\nu}$ *in* (16) *holds a.e. on* Γ.

The weak solution to Problem 1

A fivefold $(\boldsymbol{u}_1, p_1, \mathbb{S}, \boldsymbol{u}_2, \varphi)$ which satisfies

$$-\operatorname{Div}\mathbb{S}(\mathbb{D}\boldsymbol{u}_1) + \operatorname{Div}(\boldsymbol{u}_1 \otimes \boldsymbol{u}_1) + \nabla p_1 = \boldsymbol{f} \quad \text{in} \quad \Omega_1$$

(the conservation law) and

$$\boldsymbol{u}_2 = -K\nabla\varphi \quad \text{in} \quad \Omega_2$$

(the Darcy velocity), and the NSD hemivariational inequality for the Stokes velocity $\boldsymbol{u}_1 \in V_S$ and the Darcy pressure $\varphi \in P_D$, is called a weak solution of Problem 1.

Moreover, by recovering the Stokes pressure, we obtain $p_1 \in L^2_0(\Omega_1)$. Next, by $\boldsymbol{u}_2 = -K\nabla\varphi$ in Ω_2, we get the regularity $\boldsymbol{u}_2 \in L^2(\Omega_1; \mathbb{M}^d)$. From the conservation law, we obtain that $\mathbb{S}(\mathbb{D}\boldsymbol{u}) \in L^2(\Omega_1; \mathbb{M}^d)$ with $\operatorname{Div}\mathbb{S}(\mathbb{D}\boldsymbol{u}) \in L^2(\Omega_1; \mathbb{R}^d)$.

Conclusion: the weak solution to Problem 1 satisfies

$$(\boldsymbol{u}_1, p_1, \mathbb{S}, \boldsymbol{u}_2, \varphi) \in V_S \times L^2_0(\Omega_1) \times L^2(\Omega_1; \mathbb{M}^d) \times L^2(\Omega_2; \mathbb{R}^d) \times P_D.$$

A variational inequality

Corollary

Let the hypotheses of the existence theorem hold and

$$j_\tau(\boldsymbol{x},\cdot) \ \text{ be a convex function for a.e. } \ \boldsymbol{x}\in\Gamma_1.$$

Then, the hemivariational inequality in Problem 2 is equivalent to the variational inequality of the form: Find the velocity $\boldsymbol{u}_1\colon \Omega_1\to\mathbb{R}^d$ and the pressure $\varphi\colon \Omega_2\to\mathbb{R}$ such that

$$\begin{aligned}
&\int_{\Omega_1}\mathbb{S}(\mathbb{D}\boldsymbol{u}_1):\mathbb{D}(\boldsymbol{v}-\boldsymbol{u}_1)\,dx + b(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}-\boldsymbol{u}_1) + \int_{\Omega_2} K\nabla\varphi\cdot\nabla\psi\,dx\\
&+\int_{\Gamma_1} k(\boldsymbol{u}_{1\tau})(j_\tau(\boldsymbol{v}_\tau)-j_\tau(\boldsymbol{u}_{1\tau}))\,d\Gamma + \int_\Gamma\left(\varphi-\frac{1}{2}(\boldsymbol{u}_1\cdot\boldsymbol{u}_1)\right)(v_\nu-u_\nu)\,d\Gamma\\
&\quad+\frac{1}{\alpha}\int_\Gamma(\boldsymbol{u}_1\cdot\tau)\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_{1\tau})\,d\Gamma - \int_\Gamma u_{1\nu}\,\psi\,d\Gamma \ge \int_{\Omega_1}\boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}_{1\tau})\,dx\\
&\quad+\int_{\Omega_2} g\,\psi\,dx \ \text{ for all } \ (\boldsymbol{v},\psi)\in V = V_S\times P_D,
\end{aligned}$$

and this variational inequality has a solution.

Example of constitutive function

Example

Various constitutive functions $\mathbb{S}$ that satisfy the hypothesis $H(\mathbb{S})$ can be found in the literature, e.g., J. Malek, Necas, Rokyta, Ruzicka, Rajagopal, among others. For instance, the constitutive function $\mathbb{S}\colon \Omega \times \mathbb{M}^d \to \mathbb{M}^d$ defined by

$$\mathbb{S}(\boldsymbol{x}, \mathbb{D}) = \nu(\|\mathbb{D}\|)\,\mathbb{D} \quad \text{for} \quad \mathbb{D} \in \mathbb{M}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega_1,$$

where $\nu\colon [0, \infty) \to \mathbb{R}$ is a given function describes the generalized Newtonian fluids. If $\nu(s) = \nu_0$ for $s \geq 0$ with $\nu_0 > 0$ a given viscosity constant, then we have the function $\mathbb{S}(\boldsymbol{x}, \mathbb{D}) = \nu_0\,\mathbb{D}$ which is the linear law for the usual Newtonian fluid.

Examples of the slip boundary condition

Example

$$-\boldsymbol{\tau}_\tau(\boldsymbol{u}) \in k(\boldsymbol{u}_\tau)\,\partial j_\tau(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_1 \tag{18}$$

The first two examples use convex potentials.

(1) Let $k = k(\boldsymbol{x})$ and $j_\tau(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2} k(\boldsymbol{x}) \|\boldsymbol{\xi}\|^2$. Then (18) reduces to the classical Navier slip condition (1823) of the form $-\boldsymbol{\tau}_\tau(\boldsymbol{u}) = k(\boldsymbol{x})\,\boldsymbol{u}_\tau$ on Γ_1 with an adhesive coefficient k which on a nonhomogeneous surface depends on the position $\boldsymbol{x}$.

(2) Let $k = k(\boldsymbol{x})$ and $j_\tau(\boldsymbol{x}, \boldsymbol{\xi}) = k(\boldsymbol{x})\,\|\boldsymbol{\xi}\|$. Then (18) takes the form

$$\|\boldsymbol{\tau}_\tau(\boldsymbol{u})\| \leq k(\boldsymbol{x}), \quad -\boldsymbol{\tau}_\tau(\boldsymbol{u}) = k(\boldsymbol{x})\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq 0,$$

(Tresca friction law), where the modulus of friction k is assumed to be a continuous and strictly positive function. The hypothesis $H(j_\tau)$ holds with $b_2 = 0$, so, in this case the smallness condition holds trivially.

Examples of the slip boundary condition

Example

The third example exploits a nonconvex superpotential.
(3) Let $k = 1$, for simplicity, and

$$j_\tau(\boldsymbol{\xi}) = \int_0^{\|\boldsymbol{\xi}\|} \mu(s)\, ds,$$

where $\mu\colon [0,\infty) \to \mathbb{R}$ is continuous and $\mu(0) > 0$. Then (18) is equivalent to a nonmonotone law of the form

$$\|\boldsymbol{\tau}_\tau(\boldsymbol{u})\| \le \mu(0), \text{ if } \boldsymbol{u}_\tau = 0, \qquad -\boldsymbol{\tau}_\tau(\boldsymbol{u}) = \mu(\|\boldsymbol{u}_\tau\|)\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq 0.$$

For instance, $\mu(s) = (a-b)\exp(-\alpha\, s) + b$, where $a > b$ and α are given non-negative constants.

Open problems

- Identify hypotheses under which the solution to the mixed Navier-Stokes-Darcy (NSD) model is uniquely solvable.
- Continuous dependence of the solution on the given data and prove the stability of the solution set under suitable perturbations.
- Explore the NSD model for Bingham-type fluids, where the weak formulation leads to a variational-hemivariational inequality.
- Introduce additional constraints on the solution in the model, resulting in a NSD variational-hemivariational inequality with constraints.
- A parabolic-elliptic Navier-Stokes-Darcy model.
- A parabolic-parabolic Navier-Stokes-Darcy model.

References on the SD and NSD models

• G. Beavers, D. Joseph, Boundary conditions at a naturally impermeable wall, *J. Fluid Mech.* **30** (1967), 197–207.

• Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang, W. Zhao, Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions, *SIAM J. Numer. Anal.* **47** (2010), 4239–4256.

• A. Cesmelioglu, V. Girault, B. Riviére, Time-dependent coupling of Navier-Stokes and Darcy flows, *ESAIM: Math. Model. Numer. Anal.* **47** (2013), 539–554.

• M. Cui, W. Dong, Z. Guo, Global well-posedness of coupled Navier-Stokes and Darcy equations, *J. Diff. Equ.* **388** (2024), 82–111.

References on the SD and NSD models

• M. Discacciati, A. Quarteroni, Navier-Stokes-Darcy coupling: modeling, analysis, and numerical approximation, *Rev. Mat. Complut.* **22** (2009), 315–426.

• S. Pasquier, M. Quintard, Y. Davit, Modeling flow in porous media with rough surfaces: effective slip boundary conditions and application to structured packings, *Chemical Eng. Sci.* **165** (2017), 131–146.

• X. Wang, X. Guo, X. Li, A class of new linear, efficient and high-order implicit-explicit methods for the unsteady Navier-Stokes-Darcy model based on nonlinear Lions interface condition, *J. Scient. Comput.* **101** (2024), 40.

• U. Wilbrandt, *Stokes-Darcy Equations, Analytic and Numerical Analysis*, Springer Nature, Switzerland AG, 2019.

• J. Zhao, Z. Lv, Y. Xu, A weak Galerkin method for the coupled Darcy-Stokes problem with a nonstandard transmission condition on the Stokes boundary, *East Asian J. Appl. Math.* **15** (2025), 29–52.

Related references

• C. Fang, W. Han, Well-posedness and optimal control of a hemivariational inequality for nonstationary Stokes fluid flow, *Discrete Contin. Dyn. Syst.* **36** (2016), 5369–5386.

• W. Han, *An Introduction to Theory and Applications of Stationary Variational-Hemivariational Inequalities*, Springer, Cham, 2024.

• W. Han, K. Czuprynski, F. Jing, Mixed finite element method for a hemivariational inequality of stationary Navier-Stokes equations, *J. Sci. Comput.* **89** (2021), 8.

• S. Migórski, S. Dudek, A quasi-variational-hemivariational inequality for incompressible Navier-Stokes system with Bingham fluid, *Set-Valued and Variational Analysis* **32** (2024), 14.

• Y. Qian, F. Wang, Y. Zhang, W. Han, A mixed discontinuous Galerkin method for an unsteady incompressible Darcy equation, *Applicable Analysis* **101** (2022), 1176–1198.

• N. Saito, H. Fujita, Regularity of solutions to the Stokes equation under a certain nonlinear boundary condition, *Lecture Notes Pure Appl. Math.* **223** (2001), 73–86.

Acknowledgement

Supported by the National Science Centre of Poland
under Grant No. 2021/41/B/ST1/01636

stanislaw.migorski@uj.edu.pl

S. Migórski, S. Dudek, A Navier-Stokes-Darcy hemivariational inequality of elliptic type, *Nonlinear Analysis: Real World Applications* **92** (2026), 104627.