Ordered Best Proximity Pairs for Asymptotic Pointwise Noncyclic Contractions with Applications to Volterra and Matrix Equations

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Outline

- Banach spaces and contractions
- On pointwise contractions and Banach spaces
- Asymptotic pointwise contractions on Banach spaces
- Best proximity pair for noncyclic mappings in Banach spaces
- Uniformly convex Banach spaces
- Ordered proximal noncyclic contractions on uniformly convex Banach spaces
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Definition

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- $||Tx - Ty|| \leq \alpha ||x - y||, x, y \in E$.
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- Financial markets
- Matrix Theory
- Solutions of Differential and integral Equations
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On pointwise contractions in Banach spaces

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Definition[Kirk, 1970]

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We recall the well known (on pointwise contractions, Kirk's fixed point Theorem) Theorem[W A Kirk, 1970]: Let E be a Banach space and M be a nonempty subset of E. Every pointwise contraction on M has a unique fixed point. Moreover, for each $x_0 \in M$, the sequence (Picard iteration) defined by $x_{n+1} = Tx_n$ converges strongly to the fixed point of T.

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Example

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Let E be the set of real numbers equipped with $||.|| : E \to [0, \infty)$ defined by $||x|| = |x|$, then E is a Banach space. Set $M = [0, 1]$, in E and define T on M by

$$T(x) = \frac{x}{1 + x}$$

for all $x \in M$. Then we have:

- T is not a contraction on M, so we cannot use the Banach contraction Theorem to investigate the existence of a fixed point(s) for T.

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- In this case, we have T is a pointwise contraction on M, and has a unique fixed point.
- We can also show easily that the Picard iteration $x_{n+1} = Tx_n$, converges strongly to the fixed point.

Kirk's result has extended the class of maps for which fixed point theorems can be investigated and obtained in nonlinear analysis.

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On asymptotic pointwise contractions in Banach spaces

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Definition Kirk et al, 2008

Let E be a Banach space and M be a nonempty subset of E. A mapping $T : M \to M$ is an **asymptotic pointwise contraction** on M if for all $x, y \in M$ there exists a sequence of functions $\alpha_n : M \to [0, 1)$ for $n = 1, 2, 3, ...$ and a function $\alpha : M \to [0, 1)$ such that the following condition hold:

- $||T^n x - T^n y|| \leq \alpha_n(x)||x - y||, x, y \in M.$
- $\alpha_n(x) \to \alpha(x)$, as $n \to \infty$.

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Note that:

- The required contractive behavior appears on the iterates of T and not on T directly, there are situations where T may not necessarily be a contraction. The following result also increases substantially classes of maps on which fixed point results can be studied and obtained.
- We recall the well known (on asymptotic pointwise contractions, Kirk et al 2008) Theorem albeit for weakly compact subsets of a Banach space: Let E be a Banach space and M be a nonempty subset of E. Every asymptotic pointwise contraction on M has a unique fixed point. Moreover, for each $x_0 \in M$, the sequence (Picard iteration) defined by $x_{n+1} = Tx_n$ converges strongly to the fixed point of T.

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On best proximity pairs and noncyclic contractions in Banach spaces

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A Eldred,2005

Let M and N be two nonempty subsets of a Banach space E. A mapping $T : M \cup N \to M \cup N$, is **noncyclic** if $T(M) \subseteq M$ and $T(N) \subseteq N$. Then mapping T is a **noncyclic contraction** if T is noncyclic and for some $\alpha \in (0, 1)$ and all $(x, y) \in M \times N$ the following condition hold:
$||Tx - Ty|| \leq \alpha||x - y|| + (1 - \alpha)dist(M, N)$, where
$dist(M, N) = inf\{||x - y||\}$.

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We recall

For two subsets, call it a pair (M, N) in a Banach space E we define

$$M_0 = \{x \in M : ||x - \tilde{y}|| = dist(M, N)\}$$

for some $\tilde{y} \in N$. Similarly, we denote

$$N_0 = \{y \in N : ||\tilde{x} - y|| = dist(M, N)\}$$

for some $\tilde{x} \in M$. We shall say a **proximal pair** of (M, N) is the pair (M_0, N_0). If $M = M_0$, and $N = N_0$ we say that a pair (M, N) is proximal. Assume that $T : M \cup N \to M \cup N$, is a **noncyclic** mapping. A point $(x, y) \in M \times N$ is called a **best proximity pair** if $x = Tx$, and $y = Ty, ||x - y|| = dist(M, N)$.

Remark

- The concept of best pair arises naturally when the sets M and N are disjoin, so that the common fixed point in $M \cap N$ does not occur, in such a situation one seeks points in M and N that are close as possible to each other while remaining fixed under the action of T on their respective sets.
- The case when $M \cap N \neq \emptyset$, then $dist(M, N) = 0$. Any common fixed point of T becomes, a best proximity pair
- The theory of best proximity pairs is a natural extension of classical fixed point theory to the case of disjoint sets.
- In the noncyclic setting, the mapping T preserves each set individually and therefore the problem reduces to finding fixed points in M and N, that realise the minimal distance between the two sets.
- The existence of such pairs requires geometric assumptions on the Banach space E.

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Uniformly convex Banach spaces

- The Banach space E is said to be **uniformly convex** if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $x, y \in E$ satisfying $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \epsilon$, we have

$$||\frac{x+y}{2}|| \leq 1 - \delta(\epsilon).$$

- Hilbert spaces, the function space $L^p(\Omega)$ for $1 < p < \infty$, are classical examples.
- If (M, N) is nonempty, closed and a convex pair in a uniformly convex Banach space E such that either M or N is bounded, then the pair (M_0, N_0) is nonempty.

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- If M is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space E, then the **projection operator** $\mathcal{P}_M : E \to 2^M$, defined by

$$\mathcal{P}_M(x) = \{y \in M : ||x - y|| = dist(\{x\}, M)\}$$

  is single valued.
- The projection operator is important for our work, its properties are well-known in the literature.
- For our purposes, consider a uniformly convex Banach space E, and a pair (M, N) of nonempty, bounded, closed and convex subsets with M_0 nonempty. We define $\mathcal{P} : M_0 \cup N_0 \to M_0 \cup N_0$ by $\mathcal{P}(x) = \mathcal{P}_{M_0}(x)$, for $x \in N_0$, and $\mathcal{P}(x) = \mathcal{P}_{N_0}(x)$, if $x \in M_0$.
- $||x - \mathcal{P}x|| = dist(M, N)$, and $\mathcal{P}$ is cyclic.
- The restriction of $\mathcal{P}$ on M_0 and N_0, is an isometry, respectively, continuous and affine mapping
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  is single valued.
- The projection operator is important for our work, its properties are well-known in the literature.
- For our purposes, consider a uniformly convex Banach space E, and a pair (M, N) of nonempty, bounded, closed and convex subsets with M_0 nonempty. We define $\mathcal{P} : M_0 \cup N_0 \to M_0 \cup N_0$ by $\mathcal{P}(x) = \mathcal{P}_{M_0}(x)$, for $x \in N_0$, and $\mathcal{P}(x) = \mathcal{P}_{N_0}(x)$, if $x \in M_0$.
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Ordered proximal noncyclic contractions contractions

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Let E be a uniformly convex Banach space equipped with a partial order relation " $\preceq$ " and let (M, N) be a nonempty, closed and convex pair in E such that M is bounded. A mapping $T : M \cup N \to M \cup N$ is said to be an **ordered proximal noncyclic contraction** if T is noncyclic the following holds:

- $||Tx - Ty|| \leq \alpha||x - y|| + (1 - \alpha)dist(M, N)$.
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Theorem, Gabeleh and Moshokoa, 2026

Let E be a uniformly convex Banach space equipped with a partial order " $\preceq$ " and let (M, N) be a nonempty, closed and convex pair in E such that M is bounded. Suppose that $T : M \cup N \to M \cup N$ is an ordered proximal noncyclic contraction such that $T|_{M_0}$ is nondecreasing. If there exists $x_0 \in M_0$ with $x_0 \preceq Tx_0$, then T has a best proximity pair and the sequence defined by $x_n = T^n x_0, y_n = \mathcal{P}x_n$, converges to the best proximity pair of T.

- The proximity pair ensured, is not unique, actually an ordered proximal noncyclic contraction may have infinitely many proximity pairs.
- as a Corollary: Let (M, N) be a nonempty closed convex pair in a uniformly convex Banach space E, and let $T : M \cup N \to M \cup N$ be a noncyclic contraction mapping. Assume that $x_0 \in M$, and define the sequence $\{(x_n, y_n)\} \in M_0 \times N_0$ by $x_n = T^n x_0, y_n = \mathcal{P} x_n$, for $n = 1, 2, 3, \ldots$ then $\{(x_n, y_n)\}$ converges to the best proximity pair of T.

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Example, application of the theorem to ordered proximal noncyclic contractions

- Let E be $\mathcal{R}^2$ equipped with the usual norm $||.||$, for $x = (a_1, a_2)$ and $y = (b_1, b_2)$ in E, denote $x \preceq y$, iff $a_1 \leq b_1$ and $a_2 = b_2$.
- For $0 \leq t \leq 1,,$ we put $M = \{(0, t)\}$ and $N = (1, t)$, we have $dist(M, N) = 1,,$ also $M = M_0$, and $N_0 = N$.
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Ordered proximal asymptotic noncyclic contractions

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Theorem, Gabeleh and Moshokoa, 2026

Let E be a uniformly convex Banach space equipped with a partial order " $\preceq$ " and let (M, N) be a nonempty, closed and convex pair in E such that M is bounded. Suppose that $T : M \cup N \to M \cup N$ is an ordered proximal asymptotic noncyclic contraction such that $T|_{M_0}$ is nondecreasing. If there exists $x_0 \in M_0$ with $x_0 \preceq Tx_0$, then T has a best proximity pair and the sequence defined by $x_n = T^n x_0, y_n = \mathcal{P}x_n$, converges to the best proximity pair of T.

Corollary, Gabeleh and Moshokoa, 2026

Let E be a uniformly convex Banach space equipped with a partial order " $\preceq$ " such that order intervals are closed and let M be a nonempty, bounded, closed and convex subset in E. Suppose that $T : M \to M$ is a nondecreasing mapping such that there exists a sequence of functions $\alpha_n : M \to [0, 1)$, converging pointwise to a function $\alpha : M \to [0, 1)$, and

$$||T^n x - T^n y|| \leq \alpha_n(y)||x - y||$$

, for $n = 1, 2, 3, ...$ whenever $x, y \in M$ are comparable with $x \preceq y$. If there exists $x_0 \in M$ such that $x_0 \preceq Tx_0$, then T admits a fixed point $p \in M$, moreover, the Picard relation

$$x_{n+1} = Tx_n,$$

converges strongly to p.

Corollary, Gabeleh and Moshokoa, 2026

Let E be a uniformly convex Banach space and M be a nonempty, bounded, closed and convex subset in E. Suppose that $T : M \to M$ is an asymptotic pointwise contraction mapping. Then for any $x_0 \in M$ the Picard relation

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converges strongly to the unique fixed point of T.

Applications to Volterra equations

Let $E = L^p([0, R])$, be equipped with the norm

$$||x||_p = (\int_{[0,R]} |x(t)|^p dt)^{\frac{1}{p}}.$$

- It is well known that E is a uniformly convex Banach space.
- Assume that $f \in E$ and let $k : [0, R] \times [0, R] \to \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers and for $s \leq t$ in $[0, R]$, we have

$$|k(s, t)| \leq K,$$

for some positive constant K.

- Let $\phi : \mathcal{R} \to \mathcal{R}$ be continuously differentiable such

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for all real r and some positive constants a and b.

- Define $T : E \to E$, by

$$Tx(t) = f(t) + \int_{[0,R]} k(t,s)\phi(x(s))ds.$$

- let $\beta > 0$ and define $M = \{x \in E : ||x||_p \leq \beta\}$, then M is a nonempty, closed and convex.

- For $\beta KR < 1$ with

$$\beta = \frac{||f||_p + KaR^{1+\frac{1}{p}}}{1 - KbR}.$$

- The operator T is an asymptotic contraction, when $KbR < 1$. Consequently for any $x_0 \in M$, the Picard iteration

$$x_{n+1} = Tx_n$$

converges strongly in E to a fixed point of T, which is a solution to the **nonlinear Volterra** integral equation:

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Applications to Matrix equations

Let $E = M_n$ denote the space of all $n \times n$ complex matrices. We denote by H_n the real vector space of Hermitian matrices, let E be equipped with the Frobenius norm,

- Note that $(E.||.||_F)$ is a finite-dimensional Hilbert space, which is uniformly convex, and H_n is a closed and convex subset.
- Using the operator

$$||A||_2 = sup_{||x||=1}||Ax||,$$

and noting the spectral radius formular $\rho(A)$ for A in E.
- For A in E and Q in H_n we consider the discrete Lyapunov matrix equation:

$$X = A^*XA + Q.$$

- Let $M = H_n$, and define $T : M \to M$, by $TX = A^*XA + Q$. For $X, Y \in M$ we compute $TX - TY = A^*(X - Y)A$ to get

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Thank you

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