A Double Obstacle Formulation for Optimal Portfolio Selection with Transaction Costs

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OUTLINE

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- Computational Results.
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INTRODUCTION

Introduction

- What is Portfolio Selection? (Static Approach by Markowits)
- What happens in the market?
- Portfolio Rebalancing was born (Merton Model).
- Maintained a Constant weight
- $w^* = \frac{(r_f - \mu)}{\sigma^2(\phi - 1)}$,
  where μ represents the deterministic rate of return, σ represents the constant volatility of the asset. ϕ is the utility parameter.

Introduction

- Limitation is the transaction cost.
- How does transaction cost play a role?
- Transaction cost also plays a role in Rebalancing.

Problem Statement

- Portfolio management aims to maintain a desired allocation between available assets and, consequently, a targeted risk–return profile.
- As asset prices evolve over time, portfolio weights deviate from their desired levels, creating the need for portfolio rebalancing.
- In a frictionless market, continuous rebalancing is theoretically possible. However, in practice, proportional transaction costs make frequent trading costly.
- Investors therefore face a trade-off between maintaining a desirable portfolio allocation and avoiding excessive transaction costs.
- This motivates the determination of an optimal rebalancing strategy that identifies when it is optimal to buy, sell, or refrain from trading.

Research Questions

- How can a portfolio be optimally rebalanced in the presence of proportional transaction costs?
- How does reformulating the portfolio-rebalancing problem as a double-obstacle problem facilitate the characterisation and numerical solution of the optimal trading strategy?

Aim

- The aim of this project is to study portfolio selection and rebalancing in the presence of proportional transaction costs, and to analyse how transforming the resulting problem into a double-obstacle formulation facilitates the numerical determination of the optimal trading strategy.

Model I: Transformed-Value Formulation

Transformed-Value Formulation: Assets

- In Merton's continuous-time portfolio framework, the financial market consists of two assets: a risk-free asset and a risky asset.
- **Risk-free asset:**

$$\mathrm{d}B(t) = rB(t)\,\mathrm{d}t, \qquad t \in [0, T], \tag{1}$$

  where r denotes the constant risk-free interest rate.
- **Risky asset:**

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t), \qquad t \in [0, T], \tag{2}$$

  where μ denotes the expected rate of return of the risky asset, σ denotes its constant volatility, and $W(t)$ is a standard Brownian motion.
- **Interpretation of r and μ:** The difference $\mu - r$ represents the risk premium associated with investing in the risky asset.

Transformed-Value Formulation: Asset Accounts

- Let $x_0(t)$ and $x_1(t)$ denote the amounts invested (Risky and Risk-free).
- Their dynamics equation are

$$\mathrm{d}x_0(t) = \big(rx_0(t) - \kappa c(t)\big)\mathrm{d}t - (1+\lambda)\,\mathrm{d}L(t) + (1-\alpha)\,\mathrm{d}M(t), \tag{3a}$$

$$\mathrm{d}x_1(t) = \mu x_1(t)\,\mathrm{d}t + \sigma x_1(t)\,\mathrm{d}W(t) + \mathrm{d}L(t) - \mathrm{d}M(t). \tag{3b}$$

- Here, $c(t)$ denotes the consumption rate and κ is an indicator parameter determining whether consumption is included.
- The parameter λ denotes the proportional transaction cost incurred when purchasing the risky asset, while α denotes the proportional transaction cost incurred when selling it.
- $L(t)$ and $M(t)$ denote the cumulative amounts used to purchase and sell the risky asset, respectively.

Transformed-Value Formulation: Liquidation Wealth

- The **liquidation wealth** represents the amount of wealth available if the risky position is immediately converted into the risk-free asset after accounting for transaction costs.

$$x(t) = x_0(t) + (1-\alpha)x_1^+(t) - (1+\lambda)x_1^-(t),$$

where

$$x_1^+(t) = \max\{x_1(t), 0\}, \qquad x_1^-(t) = \max\{-x_1(t), 0\}.$$

- The liquidation wealth is important because it measures the investor's effective wealth after transaction costs and is used in defining the terminal payoff and the solvency region of the portfolio.

Transformed-Value Formulation: Stochastic Control Problem

- The investor chooses the consumption and trading strategies (c, L, M) in order to maximise expected discounted utility:

$$\sup_{(c,L,M)} \mathbb{E}\left[\int_0^T \kappa e^{-\beta t} U(c(t))\,\mathrm{d}t + e^{-\beta T} U(x(T))\right]. \tag{4}$$

- This optimisation is subject to the portfolio dynamics

$$\begin{aligned} x(t) &= x_0(t) + (1-\alpha)x_1^+(t) - (1+\lambda)x_1^-(t), \\ \mathrm{d}x_0(t) &= \big(r x_0(t) - \kappa c(t)\big)\mathrm{d}t - (1+\lambda)\,\mathrm{d}L(t) + (1-\alpha)\,\mathrm{d}M(t), \\ \mathrm{d}x_1(t) &= \mu x_1(t)\,\mathrm{d}t + \sigma x_1(t)\,\mathrm{d}W(t) + \mathrm{d}L(t) - \mathrm{d}M(t). \end{aligned}$$

Transformed-Value Formulation: Value Function

- The value function represents the maximum expected discounted utility that can be attained from time t onwards:

$$v(t, x_0, x_1) = \sup_{(c,L,M)} \mathbb{E}\left[\int_t^T e^{-\beta(s-t)} U(c(s))\, \mathrm{d}s + e^{-\beta(T-t)} U(x(T)) \right]. \tag{5}$$

- We consider the Constant Relative Risk Aversion (CRRA) utility function

$$U(c) = \frac{c^\phi}{\phi}, \qquad \phi \neq 0, \quad \phi < 1. \tag{6}$$

- The parameter ϕ determines the investor's degree of relative risk aversion, with relative risk aversion given by

$$1 - \phi.$$

Transformed-Value Formulation: HJB Representation of the Trading Regions

- **No-trading region:** when $\mathrm{d}L = 0$ and $\mathrm{d}M = 0$.(SDP)

$$0 = v_t + r x_0 v_{x_0} + \mu x_1 v_{x_1} + \frac{1}{2}\sigma^2 x_1^2 v_{x_1 x_1} - \beta v + \kappa u^*(v_{x_0}), \tag{7}$$

  where

$$u^*(v_{x_0}) := \frac{1-\phi}{\phi}\,(v_{x_0})^{\frac{\phi}{\phi-1}}\,.$$

- **Buying region:** when $\mathrm{d}L > 0$ and $\mathrm{d}M = 0$,

$$\mathcal{B}v := -(1+\lambda)v_{x_0} + v_{x_1} = 0.$$

- **Selling region:** when $\mathrm{d}L = 0$ and $\mathrm{d}M > 0$,

$$\mathcal{S}v := (1-\alpha)v_{x_0} - v_{x_1} = 0.$$

- Then we get HJB variational inequality

$$0 = \max\left\{v_t + \mathcal{L}v + \kappa u^*(v_{x_0}), \mathcal{B}v, \mathcal{S}v\right\}, \tag{8a}$$

$$v(T, x_0, x_1) = \frac{\left[x_0 + (1-\alpha)x_1^+ - (1+\lambda)x_1^-\right]^\phi}{\phi}. \tag{8b}$$

Transformed-Value Formulation: HJB Operators

- The differential operator in the no-trading region is

$$\mathcal{L}v := rx_0 v_{x_0} + \mu x_1 v_{x_1} + \frac{1}{2}\sigma^2 x_1^2 v_{x_1 x_1} - \beta v.$$

- The buying and selling operators are

$$\mathcal{B}v := -(1+\lambda)v_{x_0} + v_{x_1},$$

and

$$\mathcal{S}v := (1-\alpha)v_{x_0} - v_{x_1}.$$

- Hence, in the no-trading region,

$$\mathcal{B}v < 0, \qquad \mathcal{S}v < 0,$$

while one of the corresponding constraints becomes active at a trading boundary.

Transformed-Value Formulation: Transformations

- **Homotheticity transformation:** CRRA.

$$v(t, \rho x_0, \rho x_1) = \rho^{\phi} v(t, x_0, x_1), \qquad \phi \neq 0, \quad \phi < 1. \tag{9}$$

- Introducing the transformed state variable y, ln value function form:

$$v(t, x_0, x_1) = \rho^{-\phi} \Phi(t, y). \tag{10}$$

- **Logarithmic transformation:**

$$w(t, y) := \frac{1}{\phi} \ln\big(\phi \Phi(t, y)\big). \tag{11}$$

- After applying both transformations,(HJB variational inequality)

$$0 = \max \Bigg\{ w_t + \overline{\mathcal{L}} w + \kappa \left(\frac{1}{\phi} - 1\right) \exp\left(\frac{\phi w}{\phi - 1}\right) (1 - y w_y)^{\frac{\phi}{\phi - 1}},$$

$$(1 + \lambda y) w_y - \lambda, (\alpha y - 1) w_y - \alpha \Bigg\}, \tag{12a}$$

$$w(y, T) = \ln\left(1 - \alpha y^{+} - \lambda y^{-}\right). \tag{12b}$$

Transformed-Value FormulationTransformed Differential Operator

- The transformed differential operator is

$$\overline{\mathcal{L}}w := a_{11}\left(w_{yy} + \phi w_y^2\right) + b_1 w_y - \theta,$$

where

$$a_{11} := \frac{1}{2}\sigma^2 y^2 (1-y)^2,$$

$$b_1 := y(1-y)\left[\mu - r + \sigma^2 y(\phi - 1)\right],$$

and

$$\theta := \frac{\beta}{\phi} - \left[r + y(\mu - r) - \frac{1-\phi}{2}\sigma^2 y^2\right].$$

- These transformations reduce the original two-dimensional wealth problem (x_0, x_1) to a one-dimensional problem in the portfolio proportion y.

Transformed-Value Formulation: Regional Representation

- The transformed HJB variational inequality is

$$0 = \max\left\{ w_t + \overline{\mathcal{L}} w + \kappa\left(\frac{1}{\phi} - 1\right) \exp\left(\frac{\phi w}{\phi - 1}\right)(1 - y w_y)^{\frac{\phi}{\phi - 1}}, \right.$$
$$\left. (1 + \lambda y) w_y - \lambda, \quad (\alpha y - 1) w_y - \alpha \right\}, \tag{13a}$$

$$w(y, T) = \ln\left(1 - \alpha y^+ - \lambda y^-\right). \tag{13b}$$

- The gradient constraints imply

$$-\frac{\alpha}{1 - \alpha y} \leq w_y \leq \frac{\lambda}{1 + \lambda y}.$$

Transformed-Value Formulation: Trading Regions

- **No-trading region:**

$$-\frac{\alpha}{1-\alpha y} < w_y < \frac{\lambda}{1+\lambda y},$$

and therefore

$$-w_t - \overline{\mathcal{L}} w - \kappa \left(\frac{1}{\phi} - 1\right) \exp\left(\frac{\phi w}{\phi - 1}\right) (1 - y w_y)^{\frac{\phi}{\phi - 1}} = 0.$$

- **Buying boundary:**

$$w_y = \frac{\lambda}{1+\lambda y}.$$

- **Selling boundary:**

$$w_y = -\frac{\alpha}{1-\alpha y}.$$

- At the trading boundaries, the differential HJB operator satisfies the corresponding variational inequality.
- The terminal condition is

$$w(y, T) = \ln\left(1 - \alpha y^+ - \lambda y^-\right).$$

Model II: Gradient Formulation

Gradient Formulation

- To obtain a standard double-obstacle problem, we introduce

$$V(t,y) := w_y(t,y).$$

- The gradient constraints on w then become direct bounds on V:

$$-\frac{\alpha}{1-\alpha y} \leq V \leq \frac{\lambda}{1+\lambda y}.$$

- The resulting double-obstacle problem can be written as

$$-V_t - \mathcal{T}V - \kappa g = 0, \qquad \text{if } -\frac{\alpha}{1-\alpha y} < V < \frac{\lambda}{1+\lambda y}, \tag{14a}$$

$$-V_t - \mathcal{T}V - \kappa g \leq 0, \qquad \text{if } V = \frac{\lambda}{1+\lambda y}, \tag{14b}$$

$$-V_t - \mathcal{T}V - \kappa g \geq 0, \qquad \text{if } V = -\frac{\alpha}{1-\alpha y}. \tag{14c}$$

- The terminal condition is

$$V(y,T) = -\frac{\alpha}{1-\alpha y},$$

for $y \in [0,1]$.

Terms

- $\mathcal{T}V$

$$\begin{aligned}\mathcal{T}V :=& \frac{1}{2}\sigma^2 y^2(1-y)^2 V_{yy} + [\mu - r_f + (\phi-1)\sigma^2 y + (1-2y)\sigma^2]y(1-y)V_y \\ &+ \mu - r_f + (\phi-1)\sigma^2 y + [(1-2y)(\mu - r_f) + (2-3y)\sigma^2 y(\phi-1)]V \\ &+ \phi\sigma^2 y(1-y)V[(1-2y)V + y(1-y)V_y]\end{aligned}$$

- g

$$g := -\exp\left(\frac{\phi w}{\phi - 1}\right)(1-yV)^{\frac{1}{\phi-1}}[y(V)^2 + yV_y]$$

Numerical Scheme: Penalty Approximation

- The transformed problems contain inequality constraints associated with the buying and selling regions.
- We use a **penalty method** to replace these constrained problems by nonlinear PDEs that can be solved numerically.
- For the transformed-value formulation, the penalty terms enforce

$$-\frac{\alpha}{1-\alpha y} \leq w_y \leq \frac{\lambda}{1+\lambda y}.$$

- For the gradient formulation, they enforce

$$-\frac{\alpha}{1-\alpha y} \leq V \leq \frac{\lambda}{1+\lambda y}.$$

- A sufficiently large penalty parameter Pk forces violations of the trading constraints towards zero.

PDE with Penalty Terms

- Model I

$$-w_t - \overline{\mathcal{L}}w = Pk\Big(\big[w_y - \frac{\lambda}{(1+\lambda y)}\big]^+ + \big[-w_y - \frac{\alpha}{(1-\alpha y)}\big]^+\Big), \quad t \in [0, T], \quad y \tag{15a}$$

$$w(y, T) = \ln(1 - \alpha y^+ - \lambda y^-) \quad y \in [0, 1]. \tag{15b}$$

where $\overline{\mathcal{L}}w := a_{11}\big(w_{yy} + \phi w_y^2\big) + b_1 w_y - \theta$, $a_{11} := \frac{1}{2}\sigma^2 y^2 (1-y)^2$

- Model II

$$-V_t - \mathcal{T}V = -Pk\big[V - \frac{\lambda}{1+\lambda y}\big]^+ + Pk\big[-V - \frac{\alpha}{1-\alpha y}\big]^+. \tag{16}$$

Numerical Scheme: Finite-Difference Solution

- The penalised PDE is discretised using a finite-difference method.
- A Crank–Nicolson time-stepping scheme is used to approximate the time evolution of the solution.
- Crank–Nicolson is attractive because it is second-order accurate in time for sufficiently smooth solutions and combines information from two consecutive time levels.
- The penalty and transformed differential equations introduce nonlinear terms.
- Newton-type methods(Newton–Krylov method) is used at each time step to solve the resulting nonlinear algebraic system.
- The full discretisation is omitted here; the focus is on the resulting optimal trading regions.

Computational Results

Parameters

- The following parameter values were used:

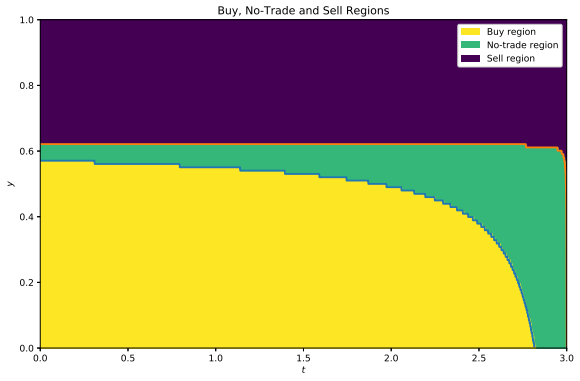
$$\lambda = 0.01, \quad \alpha = 0.01, \quad \mu = 0.18, \quad r_f = 0.07, \quad \sigma = 0.3, \quad \phi = -1,$$

$$T = 3, \quad \beta = 0.1, \quad N_y = 200, \quad N_t = 5000, \quad y \in [0, 1], \quad t \in [0, T].$$

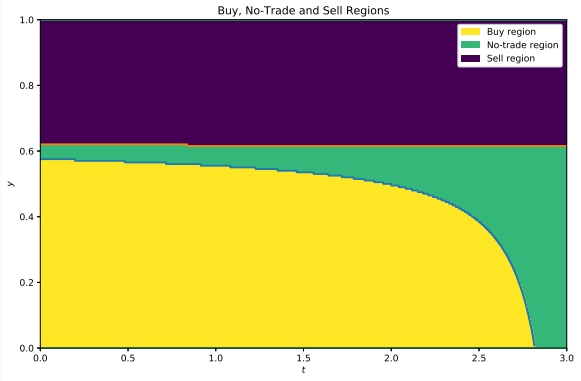
- The penalty parameter Pk is varied in order to compare Model I and Model II.

Penalty Sensitivity: Model I and Model II

$Pk = 0$

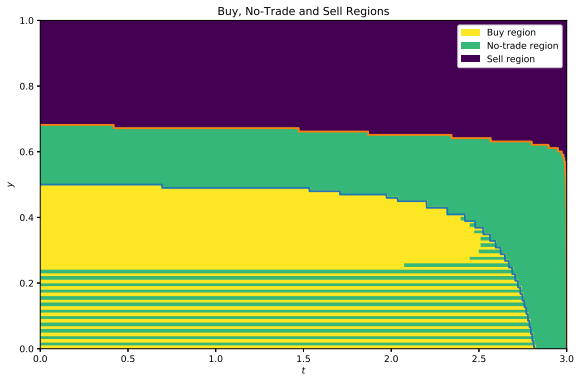


Model I

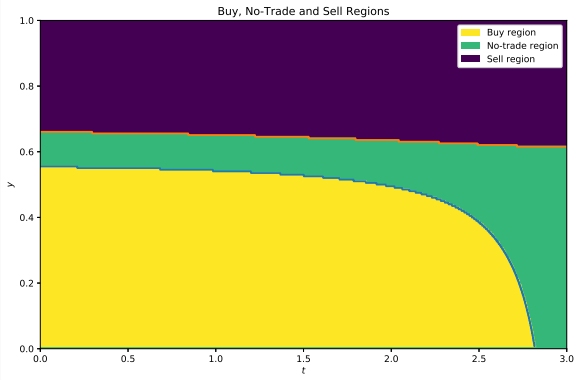


Model II

$Pk = 4$



Model I

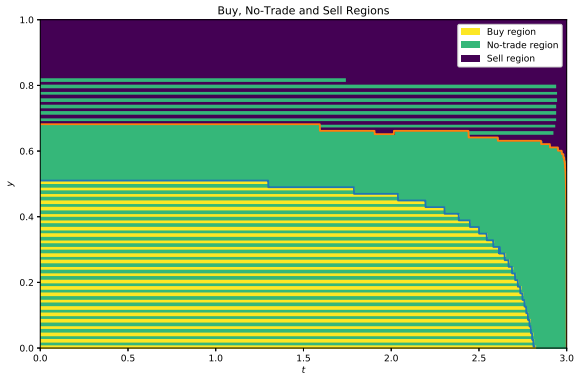


Model II

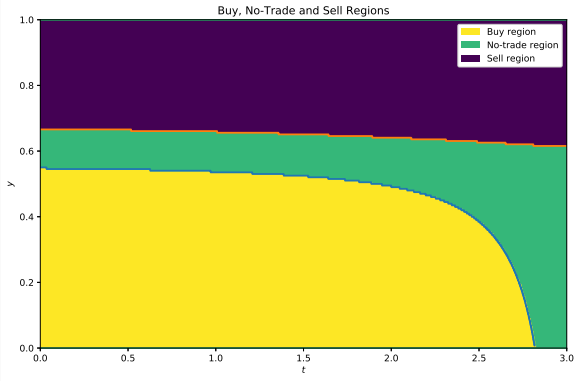
Figure 1: Comparison of the trading regions for Model I and Model II for $Pk = 0$ and $Pk = 4$.

Penalty Sensitivity: Model I and Model II Continued

$Pk = 8$

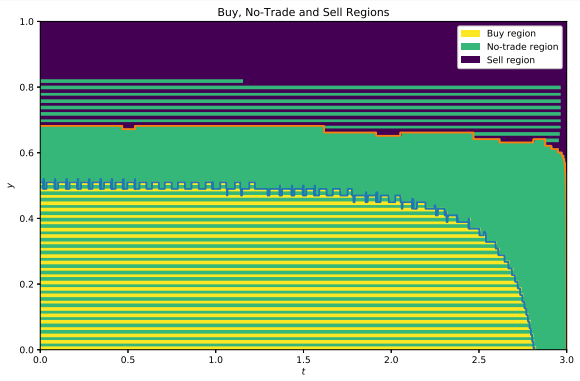


Model I

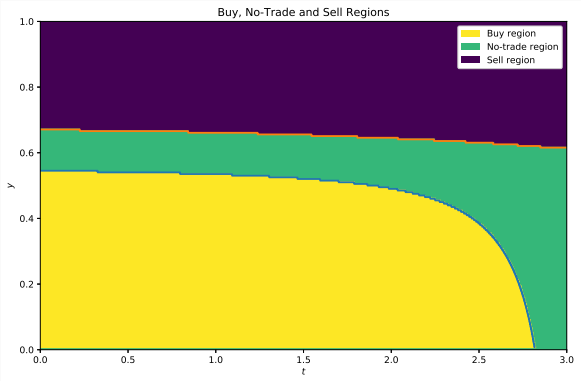


Model II

$Pk = 12$



Model I



Model II

Figure 2: Comparison of the trading regions for Model I and Model II for $Pk = 8$ and $Pk = 12$.

Observation

- The penalty parameter Pk is a numerical parameter used to enforce the transaction-cost constraints.
- Changing Pk does not change the original double-obstacle problem, but it affects the penalised approximation of that problem.
- When Pk is too small, the constraints may not be enforced sufficiently. When Pk is too large, the penalty term may dominate the numerical scheme and lead to stiffness or instability.
- In the present numerical experiments, varying Pk produces the clearest visible difference between Model I and Model II.

Conclusion and Future work

- What has been there.
- Looking at different ways of taking into account transation cost (Shadow approach, Market cost and Fixed/Mixed Costs.
- What is the focus now? Computational solution (Finite Element, asymptotic approach(analyse) or upwind Finite Scheme)

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Any questions?