Legendre transform dual asymptotic solution for insurers under the Heston Local-Stochastic Volatility Model: A comparison of variance premium and expected value principles.

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Introduction

- Reinsurance and investment are fundamental tools that enable insurers to manage claim risks, improve financial stability, and generate additional returns from surplus capital (Browne, 1995; Schmidli, 2002; Guan and Liang, 2014; Zhang et al., 2016).
- Extensive research has investigated optimal reinsurance-investment strategies under different objectives, including expected utility maximization, mean-variance optimization, and minimization of ruin probability (Hipp and Plum, 2000; Yang and Zhang, 2005).
- Most existing studies consider a **single insurer**. However, in practice insurers operate in competitive markets and often evaluate their performance relative to competitors.

Introduction cont...

Non-Zero-Sum and Zero-Sum Stochastic Games

Such interactions naturally lead to **non-zero-sum stochastic differential games** and **zero-sum stochastic differential games** involving investment and reinsurance decisions (Bensoussan and Siu, 2014; Chen et al., 2018).

Under a **non-zero-sum game**, insurance companies may achieve mutually beneficial outcomes, allowing for a potential **win–win situation**. In contrast, a **zero-sum game** assumes that one player's gain corresponds to another player's loss. Therefore, the non-zero-sum framework provides a more suitable setting for studying investment and reinsurance decisions between competing insurance companies.

Research Gap and Motivation

- Moreover, empirical evidence suggests that financial market volatility is stochastic rather than constant, motivating the use of advanced volatility models such as the Heston, Constant Elasticity of Variance (CEV), and Heston Local-Stochastic Volatility (HLSV) models (Li et al., 2012; He and coauthors, 2023).
- Most studies adopt only the **Expected Value Premium Principle** despite the practical importance of the **Variance Premium Principle** in non-life insurance.
- Limited attention has been given to competing insurers under the **Heston Local-Stochastic Volatility (HLSV)** model, which combines the advantages of both the Heston and CEV models.
- This study develops a non-zero-sum stochastic differential game for two competing insurers and compares optimal reinsurance-investment strategies under:
  - Variance Premium Principle,
  - Expected Value Premium Principle.

Model Formulation

Financial Market

- A risk-free asset:

$$dS_0(t) = rS_0(t)dt. \tag{1}$$

- A risky asset following the Heston Local-Stochastic Volatility (HLSV) model:

$$dS(t) = S(t)\Big(\mu dt + \sqrt{L(t)}S^{\beta}(t)dW_2(t)\Big), \tag{2}$$

$$dL(t) = (\Psi_1 - \Psi_2 L(t))dt + \theta\sqrt{L(t)}dW_3(t). \tag{3}$$

- The HLSV model combines the advantages of the CEV and Heston models, capturing both local and stochastic volatility effects.

Heston Local-Stochastic Volatility Model

The Heston Local-Stochastic Volatility (HLSV) model consists of two main equations.

1. Stock Price Dynamics

$$dS(t) = S(t)\left(\mu\, dt + \sqrt{L(t)}S^{\beta}(t)\, dW_2(t)\right), \qquad S(0) = s_0 > 0.$$

Explanation: This equation describes how the stock price $S(t)$ changes over time.

- The term $\mu\, dt$ represents the expected return of the stock
- $\sqrt{L(t)}S^{\beta}(t)dW_2(t)$ represents the random movement of the stock price.Here, $L(t)$ is the stochastic volatility factor.
- $S^{\beta}(t)$ is the CEV component, and $W_2(t)$ is a Brownian motion. The CEV component allows the volatility to change with the stock price, making the model more flexible than a model with constant volatility.

2. Volatility Dynamics

$$dL(t) = (\Psi_1 - \Psi_2 L(t))\, dt + \theta\sqrt{L(t)}\, dW_3(t), \qquad L(0) = l_0 > 0.$$

This equation describes the stochastic volatility component of the Heston model.

- The volatility factor $L(t)$ follows a mean-reverting square-root process.
- The term $(\Psi_1 - \Psi_2 L(t))dt$ represents the mean-reversion effect
- $\theta\sqrt{L(t)}\, dW_3(t)$ represents random fluctuations in volatility.
- Ψ_1 controls the long-term level of the volatility, Ψ_2 controls the speed of mean reversion,
- θ controls the volatility and $W_3(t)$ is a Brownian motion.

What Are These Principles?

Expected Value Principle
Under the **Expected Value Principle**, the reinsurance premium is mainly based on the expected claim amount, with a safety loading (extra amount of money charged above the expected claim) added to account for risk.

Variance Premium Principle
Under the **Variance Premium Principle**, the reinsurance premium is based on both the expected claim amount and the variability of claims. The premium therefore increases as the variability of claims increases.

In simple terms:
Both principles are methods used to determine the premium that the insurer pays to the reinsurer for accepting part of the insurance risk.

Why Use These Principles?

Expected Value Principle
This principle is simple and easy to apply and can be suitable when claim amounts are relatively stable and predictable.

Variance Premium Principle
This principle is particularly suitable for non-life insurance, where claim amounts can be uncertain and highly variable.

$$\text{Expected Value Principle} \quad \Rightarrow \quad \text{Expected Claim} + \text{Safety Loading}$$

$$\text{Variance Premium Principle} \quad \Rightarrow \quad \text{Expected Claim} + \text{Variance-Based Loading}$$

Where variance-based loading is an extra premium charged for the uncertainty or variability of claims.

Surplus Dynamics of the Variance Premium Principle

$$dU_j^{\tilde{R}}(t) = \tau\sigma_j^2(\lambda_j + \lambda)(1 - q_j^2(t))dt + (1 - q_j(t))\sqrt{\lambda_j + \lambda}\sigma_j dW_1(t). \tag{4}$$

Interpretation of the Equation

- The first term,
$$\tau\sigma_j^2(\lambda_j + \lambda)(1 - q_j^2(t))dt,$$
represents the expected growth of the insurer's surplus. It depends on the claim variance σ_j^2, claim arrival intensity $(\lambda_j + \lambda)$, and the reinsurance strategy $q_j(t)$.
- The second term,
$$(1 - q_j(t))\sqrt{\lambda_j + \lambda}\sigma_j dW_1(t),$$
captures the random fluctuations in surplus caused by claim uncertainty.
- The parameter $q_j(t)$ controls the amount of risk transferred to the reinsurer.

Wealth process under the Variance Premium Principle

The wealth process of the insurer j is obtained by combining returns from the risk-free asset, the risky asset governed by the HLSV model, and the surplus process under the variance premium principle. The insurer dynamically chooses optimal investment strategy $p_j(t)$ and the optimal reinsurance strategy $q_j(t)$ to maximize the expected utility of terminal wealth.

$$
\begin{cases}
\mathrm{d}X_j^{u_j(t)} = (X_j^{u_j(t)} - p_j(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + p_j(t)\frac{\mathrm{d}S(t)}{S(t)} + \mathrm{d}U_j^{\tilde{R}}(t), \\
= \left(X_j^{u_j(t)} r + p_j(t)(\mu - r) + \tau\sigma_j^2(\lambda_j + \lambda)(1 - q_j(t)^2)\right)dt \\
+ p_j(t)\sqrt{L(t)}S^{\beta}dW_2(t) + (1 - q_j(t))\sqrt{(\lambda_j + \lambda)}\sigma_j dW_1(t), \\
X_j^{u_j(t)}(0) = x_j > 0.
\end{cases} \tag{5}
$$

Surplus Dynamics of the Expected Value Principle

$$d\bar{U}_j(t) = \Big(\bar{a}_j\nu_j + \bar{a}_j\vartheta_j - \bar{a}_j\vartheta_j\bar{q}_j(t)\Big)dt + \bar{q}_j(t)\bar{b}_j dW_4(t). \tag{6}$$

Interpretation of the Equation

- The drift term
$$\Big(\bar{a}_j\nu_j + \bar{a}_j\vartheta_j - \bar{a}_j\vartheta_j\bar{q}_j(t)\Big)dt$$
represents the insurer's expected surplus growth. It depends on the premium loading ϑ_j, claim intensity ν_j, and the retained proportion of risk $\bar{q}_j(t)$.
- The diffusion term
$$\bar{q}_j(t)\bar{b}_j dW_4(t)$$
captures the uncertainty associated with the retained claims. The greater the retained risk, the larger the fluctuations in the insurer's surplus.

Wealth process under the expected value principle

The wealth process of the insurer j is obtained by combining returns from the risk-free asset, the risky asset governed by the HLSV model, and the surplus process under the expected value premium principle. The insurer dynamically chooses the optimal investment strategy $\bar{p}_j(t)$ and the optimal reinsurance strategy $\bar{q}_j(t)$ to maximize the expected utility of terminal wealth.

$$
\begin{cases}
\mathrm{d}\bar{X}_j^{\bar{u}_j(t)} = (\bar{X}_j^{\bar{u}_j(t)} - \bar{p}_j(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + \bar{p}_j(t)\frac{\mathrm{d}S(t)}{S(t)} + \mathrm{d}\bar{U}_j(t) \\
= \left(\bar{X}_j^{\bar{u}_j(t)} r + (\mu - r)\bar{p}_j(t) + \bar{a}_j\nu_j + \bar{a}_j\vartheta_j - \bar{a}_j\vartheta_j\bar{q}_j(t)\right)dt \\
+\bar{p}_j(t)\sqrt{L(t)}S^{\beta}dW_2(t) + \bar{q}_j b_j dW_4(t), \\
\bar{X}_j^{\bar{u}_j(t)}(0) = \bar{x}_j > 0.
\end{cases}
\tag{7}
$$

Competition of the model

- Each insurer also compares its wealth with the wealth of the other insurer. Therefore, they want to perform better than their competitor.
- In a competitive insurance market, an insurer looks at both its own financial performance and the performance of its competitor. This comparison helps the insurer understand how well it is doing in the market and can affect its market position and reputation.
- Let $0 < K_j < 1$. The parameter K_j shows how much insurer j cares about the performance of the other insurer. A higher value of K_j means that the insurer places more importance on comparing its wealth with that of its competitor. The remaining weight, $1 - K_j$, represents the importance given to the insurer's own wealth.
- For $j = 1, 2$, the relative wealth of insurer j is defined by

$$F_j^{u_j,u_k}(t) = X_j^{u_j}(t) - K_j X_k^{u_k}(t).$$

- Here, $X_j^{u_j}(t)$ is the wealth of insurer j, while $X_k^{u_k}(t)$ is the wealth of the competing insurer. Thus, $F_j^{u_j,u_k}(t)$ measures the wealth of insurer j relative to its competitor.

Competition of the model under Variance Premium Principle

To study competition between the two insurers, we consider their relative wealth. The relative wealth of insurer j is obtained by subtracting K_j times the wealth of the competing insurer from its own wealth. Therefore, the dynamics of the relative wealth process are given by

$$
\begin{cases}
dF_j^{u_j,u_k}(t) = dX_j^{u_j}(t) - K_j dX_k^{u_k}(t), \\
= \Big(rF_j^{(u_j,u_k)} + (\mu - r)(p_j(t) - K_j p_k(t)) + \tau\big[\sigma_j^2(\lambda_j + \lambda)(1 - (q_j(t))^2) - \\
K_j\sigma_k^2(\lambda_k + \lambda)(1 - (q_k(t))^2)\big] \Big) dt + \sqrt{L(t)}S^{\beta}(t))\big(p_j(t) - K_j p_k(t)\big) dW_2(t) + \\
\Big((1 - q_j(t))\sqrt{(\lambda_j + \lambda)}\sigma_j - K_j(1 - q_k(t))\sqrt{(\lambda_k + \lambda)}\sigma_k \Big) dW_1(t)
\end{cases}
$$

Competition of the model under Expected Value Principle

Under the Expected Value Principle, we also consider the relative wealth of the two insurers. The relative wealth of insurer j compares its wealth with the wealth of its competitor. The dynamics of the relative wealth process are given by

$$
\begin{cases}
d\bar{F}_j^{\bar{u}_j,\bar{u}_k}(t) = d\bar{X}_j^{\bar{u}_j}(t) - K_j d\bar{X}_k^{\bar{u}_k}(t) \\
= \Big(r\bar{F}_j^{(\bar{u}_j,\bar{u}_k)} + (\mu - r)(\bar{p}_j(t) - K_j\bar{p}_k(t)) + (\bar{a}_j\nu_j - K_j\bar{a}_k\nu_k) + \big(\bar{a}_j\vartheta_j - K_j\bar{a}_k\vartheta_k\big) - \\
\big(\bar{a}_j\vartheta_j\bar{q}_j(t) + K_j\bar{a}_k\vartheta_k\bar{q}_k(t)\big)\Big)dt + \sqrt{L(t)}S^{\beta}(t)dW_2(t))\big(\bar{p}_j(t) - K_j\bar{p}_k(t)\big) + \\
\big(\bar{q}_j(t)b_j - K_j\bar{q}_k(t)b_k\big)dW_1(t)
\end{cases}
$$

Solution of the model

Suppose both insurers want to maximize the following objective function:

$$\mathcal{B}(t, f_j, s, l) = \sup_{u_j \in \Pi_j} \mathbb{E}\left[U(F_j(T)) \,|\, S(t) = s,\, F_j(t) = f_j,\, L(t) = l\right]. \tag{8}$$

with the boundary condition

$$\mathcal{B}(T, f_j, s, l) = U(f_j). \tag{9}$$

The exponential utility function of insurer j is given by

$$U^{(j)}(f_j) = -\frac{1}{c_j} e^{-c_j f_j}, \qquad c_j > 0. \tag{10}$$

Hamilton–Jacobi–Bellman equation (HJB) Equation

Applying the dynamic programming principle to the value function $\Phi^j(t, f_j, s, I)$ gives the HJB equation

$$\boxed{\sup_{u_j \in \Pi_j} \mathcal{A}^{u_j, u_k} \Phi^j(t, f_j, s, I) = 0.} \tag{11}$$

where

$$u_j(t) = \big(p_j(t), q_j(t)\big), \qquad u_k(t) = \big(p_k(t), q_k(t)\big),$$

represent the investment and reinsurance controls of insurers j and k, respectively.

The controlled operator $\mathcal{A}^{u_j, u_k}$ incorporates:

- the drift terms of the state variables;
- the diffusion and variance terms;
- the covariance and cross-variation terms;
- the effects of investment and reinsurance decisions.

Controlled HJB Operator under Variance Premium Principle

$$\mathcal{A}^{u_j,u_k}\Phi^j(t,f_j,s,l) = \Phi^j_t + \Big[rf_j + (\mu - r)(p_j(t) - K_j p_k(t)) + \tau\big[\sigma_j^2(\lambda_j + \lambda)(1 - (q_j(t))^2) -$$

$$K_j\sigma_k^2(\lambda_k + \lambda)(1 - (q_k(t))^2)\big]\Big]\Phi^j f_j + (\Psi_1 - \Psi_2 l)\Phi^j_l + \mu s\Phi^j_s +$$

$$\frac{1}{2}\Big[l(t)s^{2\beta}p_j^2(t) - 2l(t)s^{2\beta}K_j p_j(t)p_k(t) + l(t)s^{2\beta}K_j^2 p_k^2(t) + (1 - q_j(t))^2(\lambda_j + \lambda)\sigma_j^2 -$$

$$2K_j(1 - q_j(t))(1 - q_k(t))\sqrt{(\lambda_j + \lambda)(\lambda_k + \lambda)}\sigma_j\sigma_k + K_j^2(1 - q_k(t))^2(\lambda_k + \lambda)\sigma_k^2\Big]\Phi^j_{ff} +$$

$$\frac{1}{2}\big[\theta^2 l(t)\big]\Phi^j_{ll} + \frac{1}{2}l(t)s^{2\beta+2}\Phi^j_{ss} + \Big[p_j(t)\theta l(t)s^{\beta}\rho_{23} - K_j\theta l(t)p_k(t)s^{\beta}\rho_{23}\Big]\Phi^j_{fl} +$$

$$\Big[l(t)s^{2\beta+1}p_j(t) - K_j l(t)s^{2\beta+1}p_k(t)\Big]\psi^j_{f_j s} + \theta l(t)s^{\beta+1}\rho_{23}\Phi^j_{sl}.$$

Controlled HJB Operator under Expected Value Principle

$$\bar{\mathcal{A}}^{\bar{u}_j,\bar{u}_k} = \psi_t + \Big[r\bar{f}_j + (\mu - r)(\bar{p}_j(t) - K_j\bar{p}_k(t)) + (\bar{a}_j\nu_j - K_j\bar{a}_k\nu_k) + (\bar{a}_j\vartheta_j - K_j\bar{a}_k\vartheta_k) -$$

$$\big(\bar{a}_j\vartheta_j\bar{q}_j(t) + K_j\bar{a}_k\vartheta_k\bar{q}_k(t)\big)\Big]\psi^j_{\bar{f}_j} + \mu s\psi^j_s + (\Psi_1 - \Psi_2 l(t))\psi^j_l +$$

$$\frac{1}{2}\Big[l(t)s^{2\beta}\bar{p}^2_j(t) - 2K_j l(t)s^{2\beta}\bar{p}_j(t)\bar{p}_k(t) + K_j^2 l(t)s^{2\beta}\bar{p}^2_k(t) + \bar{q}^2_j(t)b^2_j -$$

$$2K_j\bar{q}_j(t)\bar{q}_k(t)b_j b_k + K_j^2\bar{q}^2_k(t)b^2_k\Big]\psi^j_{\bar{f}_j\bar{f}_j} + \frac{1}{2}l(t)s^{2\beta+2}\psi^j_{ss}$$

$$+ \frac{1}{2}\theta^2 l(t)\psi^j_{ll} + \Big[l(t)s^{2\beta+1}\bar{p}_j(t) - K_j l(t)s^{2\beta+1}\bar{p}_k(t)\Big]\psi^j_{\bar{f}_j s} +$$

$$\theta l(t)s^{\beta}\rho_{23}\big[\bar{p}_j(t) - K_j\bar{p}_k(t)\big]\psi^j_{\bar{f}_j l} + \theta l(t)s^{\beta+1}\rho_{23}\psi^j_{ls}$$

Why Legendre Transformation?

The HJB equation contains nonlinear terms involving the investment and reinsurance controls p_j, p_k, q_j and q_k.

These nonlinear control terms make the direct optimization of the HJB equation difficult.

Therefore, a **Legendre transformation** is introduced to transform the nonlinear optimization problem into a more tractable form.

Purpose:

- simplify the nonlinear terms in the HJB equation;
- facilitate optimization with respect to the control variables;
- obtain explicit expressions for the optimal investment and reinsurance strategies.

Why Legendre Transformation?

Primal Optimization Problem: First, we formulate the original optimization problem. The insurer chooses the best strategy u to maximize the expected utility of terminal relative wealth

$$G(t, s, l, f) = \sup_{u} \mathbb{E}\Big[U(F(T))\Big],$$

where f denotes the relative wealth process.

Legendre Transformation: Because this problem is difficult to solve directly in terms of relative wealth f, we apply the Legendre transformation. This changes the problem to a dual form using the new variable z, which helps us simplify the solution.

$$\widehat{G}(t, s, l, z) = \sup_{f} \{G(t, s, l, f) - zf\}.$$

The dual variable is defined by the derivative of the value function with respect to relative wealth:

$$z = G_f(t, s, l, f).$$

Optimal Investment Strategy: Variance Premium Principle

$$p_j^*(t) = \pi_t^*(t) + K_j p_k^*(t)$$

The optimal investment consists of two components:

- $\pi_t^*(t)$ represents the insurer's **own optimal investment decision**, based on its risk and return considerations.
- $K_j p_k^*(t)$ represents an **adjustment for the competitor's investment decision**.

Thus, the insurer j considers both its own investment opportunities and the investment behavior of insurer k.

Optimal Reinsurance Strategy: Variance Premium Principle

$$q_j^*(t) = \frac{c}{2\tau e^{r_0(T-t)} + c}\left[1 - \frac{K_j(1-q_k^*(t))\sqrt{\lambda_k+\lambda}\sigma_k}{\sigma_j\sqrt{\lambda_j+\lambda}}\right].$$

The optimal reinsurance decision depends on:

- the insurer's claim volatility σ_j and claim arrival intensity $(\lambda_j + \lambda)$.
- the competitor's reinsurance decision $q_k^*(t)$;
- the relative performance parameter K_j;
- the insurer's risk-aversion parameter c and the variance premium parameter τ.

Optimal Strategies: Expected Value Principle

Optimal Investment Strategy:

$$\bar{p}_j^*(t) = p_j^*(t)$$

This means that the optimal investment strategy is unchanged under the expected value principle. Therefore, insurer j makes the same investment decision as under the variance premium principle.

Optimal Reinsurance Strategy:

$$\bar{q}_j^*(t) = -\frac{\bar{a}_j \vartheta_j e^{-r_0(T-t)}}{c\bar{b}_j^2} + \frac{K_j \bar{q}_k^*(t)\bar{b}_k}{\bar{b}_j}.$$

In simple terms, the optimal reinsurance decision depends on:

- the insurer's own risk characteristics and the premium loading ϑ_j;
- the insurer's risk-aversion parameter c and the remaining time to maturity;
- the competitor's reinsurance decision $\bar{q}_k^*(t)$;
- the relative performance parameter K_j, which captures the effect of competition.

Main Economic Insights

Identical Investment Strategies

Identical Investment Strategies:
The optimal investment strategies are identical under both premium principles, as they are determined solely by financial market dynamics.

Economic Interpretation: Insurers face the same investment opportunities and market risks regardless of the method used to calculate reinsurance premiums.

Main Economic Insights

Separation of Decisions

Separation of Decisions:
Investment strategies are primarily influenced by **financial market factors**, while reinsurance strategies are mainly determined by **insurance market factors**.

Economic Interpretation: Investment and reinsurance decisions respond to different sources of risk and can therefore be considered separately in the decision-making process.

Main Economic Insights cont...

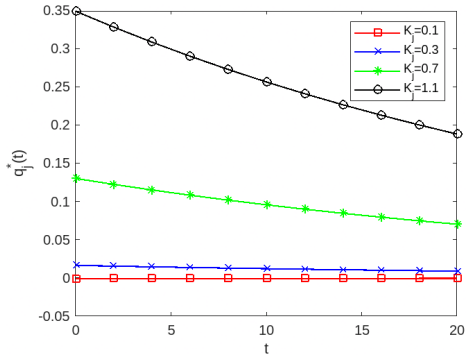
Different Reinsurance Strategies

For the **expected value principle**, the reinsurance premium is mainly based on the expected claim amount, while the variability of claims is not considered.
In contrast, under the **variance premium principle**, the reinsurance premium considers both the expected claim amount and the variability of claims.

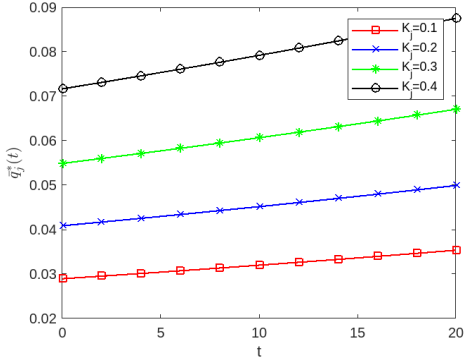
Economic Interpretation

- **Expected Value Principle:** The premium mainly reflects the **average expected loss**, making the pricing simpler and less sensitive to claim risk.
- **Variance Premium Principle:** The premium reflects both the **expected loss** and an **additional charge for claim uncertainty**. Therefore, greater claim variability results in a higher reinsurance premium.

Effect of K_j on Reinsurance Strategies



(a) Effect of K_j on $q_j^*(t)$



(b) Effect of K_j on $\bar{q}_j^*(t)$

As K_j increases, where K_j measures the strength of insurer j's concern about its performance relative to its competitor, both insurers choose higher reinsurance retention levels. Thus, stronger competition encourages insurers to retain more risk and purchase less reinsurance.

Conclusion

- The study derives optimal investment and reinsurance strategies for two competing insurers under the HLSV model.
- The **investment strategy is the same** under both premium principles and depends on financial market conditions.
- The **reinsurance strategy differs**: the variance premium principle gives more attention to claim uncertainty.
- Stronger competition leads insurers to **retain more risk and buy less reinsurance** to improve relative performance.

Thank You

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