A Study of Optimization Problems in Real Hilbert Spaces from Continuous and Discrete Perspectives

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Presentation @ MOOCTA Workshop
The Manhattan Hotel, Pretoria.

September 7, 2026.



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Introduction

Overview

As a general principle, when many options are available, humans naturally strive to make the best possible choice. In effect, everyday human actions revolve around solving optimization problems, either consciously or unconsciously.

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Overview

As a general principle, when many options are available, humans naturally strive to make the best possible choice. In effect, everyday human actions revolve around solving optimization problems, either consciously or unconsciously. An optimization problem seeks the best option from all valid choices (set of solutions) for a given task. In a Hilbert space $\mathcal{H}$, such problems can be classified as either

- constrained $(\min / \max_{x \in \mathcal{C} \subset \mathcal{H}} f(x))$
- unconstrained $(\min / \max_{x \in \mathcal{H}} f(x))$

for either a convex or nonconvex function.



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Variations of Mathematical Optimization Problems

Mathematical optimization problems arising in engineering, economics, and the natural and life sciences can be formulated as

- Fixed point problem



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- Fixed point problem
- Monotone inclusion problem



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- Fixed point problem
- Monotone inclusion problem
- Mixed variational inequality problem

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Variations of Mathematical Optimization Problems

Mathematical optimization problems arising in engineering, economics, and the natural and life sciences can be formulated as

- Fixed point problem
- Monotone inclusion problem
- Mixed variational inequality problem
- Equilibrium problem
- Saddle point problem

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Variations of Mathematical Optimization Problems

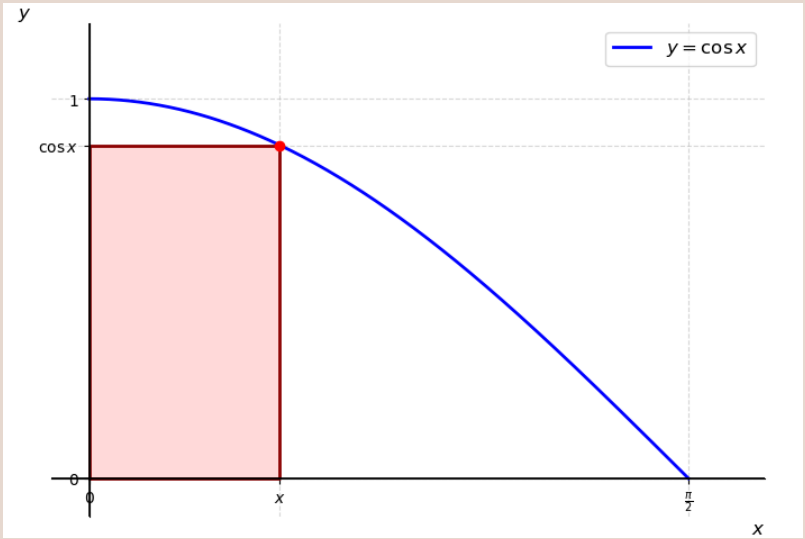
Mathematical optimization problems arising in engineering, economics, and the natural and life sciences can be formulated as

- Fixed point problem
- Monotone inclusion problem
- Mixed variational inequality problem
- Equilibrium problem
- Saddle point problem
- Feasibility problem and more

Examples

Example (Constrained Optimization Problem)

A rectangle is drawn in the first quadrant with two sides on the co-ordinate axes and one corner on the curve $y = \cos x$, where $0 \leq x \leq \frac{\pi}{2}$. Find the maximizer of the area of the rectangle. (Adapted from: MATH1048 Engineering Mathematics Calculus Study Guide: WITS University).





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$$\max_{(x,y)\in\mathbb{R}^2} A(x,y) = xy$$

$$\text{subject to} \quad y - \cos x = 0$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y \geq 0.$$

Examples

$$\max_{(x,y)\in\mathbb{R}^2} A(x,y) = xy$$

$$\text{subject to} \quad y - \cos x = 0$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y \geq 0.$$

$$x = \arctan\left(\frac{1}{x}\right) \quad \Longrightarrow \quad x \approx 0.860333589\ldots$$

is the maximizer.

Examples

Example (Unconstrained Problem)

Find $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x})$ if

$$\begin{aligned} f(\mathbf{x}) &= e^{x_1^4+x_2^2+x_3^2} + x_1^4 + x_2^2 + x_3^2 + 5x_1 \\ &\quad - \cos(x_2) + \sin(x_3) + x_1 x_2 x_3. \end{aligned}$$

Examples

Example (Unconstrained Problem)

Find $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x})$ if

$$f(\mathbf{x}) = e^{x_1^4+x_2^2+x_3^2} + x_1^4 + x_2^2 + x_3^2 + 5x_1 - \cos(x_2) + \sin(x_3) + x_1x_2x_3.$$

- Gradient descent method ($\mathbf{x}_{n+1} = \mathbf{x}_n - \lambda_n \nabla f(\mathbf{x}_n)$);

Examples

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- Gradient descent method ($\mathbf{x}_{n+1} = \mathbf{x}_n - \lambda_n \nabla f(\mathbf{x}_n)$);
- Newton's method ($\mathbf{x}_{n+1} = \mathbf{x}_n - \lambda_n [H(\mathbf{x}_n)]^{-1} \nabla f(\mathbf{x}_n)$).

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Limitations of Gradient Descent and Newton's Methods

- Require f to be continuously differentiable (C^1 or C^2). Fail to handle non-smooth function;

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Limitations of Gradient Descent and Newton's Methods

- Require f to be continuously differentiable (C^1 or C^2). Fail to handle non-smooth function;
- Newton's method fails to converge under singular or degenerate Hessians;
- Gradient descent is sensitivity to step size and Lipschitz constants (Diverges or oscillates wildly if the step size λ_n exceeds $2/L$ (where L is the Lipschitz constant of ∇f). Finding L in complex or infinite-dimensional problems is often difficult or impossible).

Description of Problems

Monotone Inclusion Problem

$\mathcal{A} : \mathcal{H} \rightrightarrows \mathcal{H}$, a set-valued operator

$$\text{find} \quad x \in \mathcal{H} \ : \ 0 \in \mathcal{A}(x).$$

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$$\text{Find} \;\; x \in \mathcal{H} \;\; : \;\; 0 \in (\mathcal{A} + \mathcal{B} + \mathcal{C})(x),$$

where

- $\mathcal{A}$ is maximally monotone
  ($\mathcal{A}$ is monotone if $\langle \mathcal{A}(x) - \mathcal{A}(y), x - y \rangle \geq 0$);

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where

- $\mathcal{A}$ is maximally monotone
  ($\mathcal{A}$ is monotone if $\langle \mathcal{A}(x) - \mathcal{A}(y), x - y \rangle \geq 0$);
- $\mathcal{B}$ is maximally monotone, single-valued, and Lipschitz continuous
  (Lipschitz continuous if
  $\|\mathcal{B}x - \mathcal{B}y\| \leq \mathcal{L}\|x - y\| \;\; \forall x, y \in \mathcal{H}, \; \mathcal{L} \geq 0$);



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- $\mathcal{C}$ is $\gamma-$cocoercive ($\gamma-$inverse strongly monotone) ($\mathcal{C}$ is $\gamma-$cocoercive if $\langle \mathcal{C}(x) - \mathcal{C}(y), x - y \rangle \geq \gamma \|\mathcal{C}(x) - \mathcal{C}(y)\|^2 \; \forall x, y \in \mathcal{H}, \gamma > 0$).

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Mixed Variational Inequality Problem

$\mathcal{A} : \mathcal{H} \to \mathcal{H}$, a single-valued operator, $g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, proper, convex, and lower semi-continuous function

$$\text{find} \;\; x \in \mathcal{H} \;\; : \quad \langle \mathcal{A}(x), y - x \rangle + g(y) - g(x) \geq 0, \quad \forall y \in \mathcal{H}.$$

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$$\text{find} \;\; x \in \mathcal{H} \;\; : \quad \langle \mathcal{A}(x), y - x \rangle + g(y) - g(x) \geq 0, \quad \forall y \in \mathcal{H}.$$

- g is said to be a proper function if its effective domain is nonempty (there exists at least one $\underline{u} \in \mathcal{H}$ such that $g(\underline{u}) < +\infty$) and $g(\underline{u}) > -\infty$ for any $\underline{u} \in \mathcal{H}$;
- g is known as a convex function if its domain is convex and for any $\underline{u}, \underline{v} \in dom\, g$, it follows that

$$g(\alpha \underline{u} + (1 - \alpha)\underline{v}) \leq \alpha g(\underline{u}) + (1 - \alpha) g(\underline{v}), \quad \forall \alpha \in [0, 1];$$



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- g is lower semi-continuous at a point $x_0 \in \mathcal{H}$ if:

$$\liminf_{\underline{u} \to x_0} g(\underline{u}) \geq g(x_0).$$

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Equilibrium Problem

$f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, a bifunction such that $f(x, x) = 0$

$$\text{find } x \in \mathcal{C} \ : \ f(x, y) \geq 0, \quad \forall y \in \mathcal{C}.$$



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Consider

$$\min_{x \in \mathbb{R}^n} f(x) + g(x), \tag{M1}$$

where f and g are closed, convex, and proper functions.



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$$0 \in (\partial f + \partial g)(x).$$



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Where

$$\partial g(x) = \{u \in \mathcal{H} : g(y) \geq g(x) + \langle u, y - x \rangle, \ \forall y \in \mathcal{H}\}$$

is called the subdifferential of g at x.

Discrete Iterative Models

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is called the subdifferential of g at x.
But

$$0 \in (\partial f + \partial g)(x) \iff x = \mathcal{J}_{\lambda(\partial f + \partial g)}(x),$$

where $\Big(\mathcal{I} + \lambda(\partial f + \partial g)\Big)^{-1} := \mathcal{J}_{\lambda(\partial f + \partial g)}$.

Discrete Iterative Models

So,

$$x = \arg\min_{x\in\mathbb{R}^n}\{f(x)+g(x)\} \iff x \in zer(\partial f + \partial g)$$
$$\iff x \in Fix\mathcal{J}_{\lambda(\partial f+\partial g)}.$$

Solving for this fixed point, we have

$$\begin{aligned} x_{n+1} &= \mathcal{J}_{\lambda(\partial f+\partial g)}(x_n) \\ &= \text{Prox}_{\lambda(f+g)}(x_n) = \arg\min_{x\in\mathbb{R}^n}\left\{f(x)+g(x)+\frac{1}{2\lambda}\|x-x_n\|^2\right\} \end{aligned}$$

(proximal point method)

[B. Martinet (Rev Fr Inform Rech Opér., 1970)]

[R. T. Rockafellar (SIAM J Control Optim., 1976)]

Discrete Iterative Models

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$= P_C(x_n)$ Projection method



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If, in addition, f is assumed to be differentiable in Problem (M1), then

$$0 \in (\nabla f + \partial g)(x),$$



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Proximal Gradient Method

$$x_{n+1} = \text{Prox}_{\lambda g}(x_n - \lambda \nabla f(x_n));$$

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Forward-Backward Splitting (FBS)

$$x_{n+1} = \mathcal{J}_{\lambda \mathcal{A}}(x_n - \lambda \mathcal{B}(x_n));$$

- $\mathcal{B}$ is $\gamma-$inverse strongly monotone ($\gamma-$cocoercive)

[P. L. Lions and B. Mercier (SIAM J. Numer. Anal., 1979)]

Discrete Iterative Models

Foward-Backward-Forward (FBF)–Tseng Method

$$x_{n+1} = \mathcal{P}_{\mathcal{X}} \circ \left[(\mathcal{I} - \lambda\mathcal{B}) \circ \mathcal{J}_{\lambda\mathcal{B}} \circ (\mathcal{I} - \lambda\mathcal{B}) + \lambda\mathcal{B}\right](x_n);$$

- Dual evaluation of operator $\mathcal{B}$
- $\mathcal{B}$ is Lipschitz or mere continuous

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Forward–Backward–Half Forward (FBHF)-Briceño-Davis

$$x_{n+1} = \mathcal{P}_{\mathcal{X}} \circ \left[(\mathcal{I} - \lambda\mathcal{B}) \circ \mathcal{J}_{\lambda\mathcal{A}} \circ (\mathcal{I} - \lambda(\mathcal{B} + \mathcal{C})) + \lambda\mathcal{B}\right](x_n).$$

- Single evaluation of the cocoercive operator $\mathcal{C}$ per iteration



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Our interest is drawn to dynamical system because of its practicability and applicability across a broad range of disciplines such as engineering, biology, economics, finance, communication technology, physics, ecology, and psychology. For instance, dynamical systems can be used to formulate a model describing the interactive behaviour between environment (humans) and robots (see [R. D. Beer 1995]). It can also be used to model and predict how system will evolve over time, including ecological population growth, epidemiological disease spread, and economical trends and decisions. Also, dynamical systems give overview and better understanding of the iterative algorithms. A dynamical system method consists of constructing a Cauchy problem (initial value problem) that possesses a unique global solution whose trajectories converge asymptotically to a solution of the original problem as time goes to infinity.



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First-order Dynamical System

$$\dot{x}(t) + x(t) + \lambda \nabla f(x(t)) = 0, \qquad x(0) = x_0.$$

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First-order Dynamical System

$$\dot{x}(t) + x(t) + \lambda \nabla f(x(t)) = 0, \qquad x(0) = x_0.$$

Second-order Dynamical System

$$\ddot{x}(t) + \frac{\beta}{t}\dot{x}(t) + \nabla f(x(t)) = 0,$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \qquad \text{(SDS)}$$

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- $\ddot{x}(t)$ (Inertial Term): Represents acceleration or physical momentum;

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- $\ddot{x}(t)$ (Inertial Term): Represents acceleration or physical momentum;
- $\frac{\beta}{t}$ is the asymptotic vanishing damping coefficient;
- $\nabla f(x(t))$ (Potential Force): The conservative force pulling the trajectory toward the minimum of f.



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Temporal Discretization of (SDS), h step size, with $\lambda = h^2$

$$\frac{1}{\lambda}\left(x_{n+1} - 2x_n + x_{n-1}\right) + \frac{\beta}{n\lambda}\left(x_n - x_{n-1}\right) + \nabla f(z_n) = 0.$$

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Temporal Discretization of (SDS), h step size, with $\lambda = h^2$

$$\frac{1}{\lambda}\left(x_{n+1} - 2x_n + x_{n-1}\right) + \frac{\beta}{n\lambda}\left(x_n - x_{n-1}\right) + \nabla f(z_n) = 0.$$

Particular instances of the iterate z_n

- Explicit: $z_n = x_n$, Heavy Ball with Friction, Polyak

$$X_{n+1} = x_n + \left(1 - \frac{\beta}{n}\right)(x_n - x_{n-1}) - \lambda \nabla f(x_n).$$

- Implicit: $z_n = x_{n+1}$, inertial proximal algorithm, Güler, Alvarez-Attouch

$$X_{n+1} = \mathrm{Prox}_{\lambda f}\left(x_n + \left(1 - \frac{\beta}{n}\right)(x_n - x_{n-1})\right).$$

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Particular instances of the iterate z_n

- Nesterov: $z_n = x_n + \left(1 - \frac{\beta}{n}\right)(x_n - x_{n-1})$, inertial gradient algorithm

$$\begin{cases} z_n = x_n + \left(1 - \frac{\beta}{n}\right)(x_n - x_{n-1}) \\ x_{n+1} = z_n - \lambda \nabla f(z_n). \end{cases}$$

- $[0, 1) \ni \theta_n := \left(1 - \frac{\beta}{n}\right)$ is the inertial factor,
- $\theta_n(x_n - x_{n-1})$ is the interial term.

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Remark

The inertial extrapolation term may fail to provide acceleration; for instance, when the trajectory generated by the sequence forms a spiral [C. Poon and J. Liang (2020)].

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Algorithm 1

FBHF algorithm with Relaxed Inertial and Correction Terms

1: Choose $\alpha \in (-1, 0]$ and $\beta \in [0, 1)$. Pick $w_{-1} = w_{-2},\ x_0,\ x_{-1} \in \mathcal{H}$ and $\varpi_0 > 0$ arbitrarily. Let $0 < \mu_1 < \mu_0 < 1,\ r \in (1, 2)$ such that $r\mu_0 \in (0, 1)$. Let $\theta_n \in (0, 1]$ and $\{\delta_n\}$ be a positive real sequence satisfy $\sum_{n=0}^{\infty} \delta_n < +\infty$, then set $n = 0$.

2: Given the current iterates ϖ_n, w_{n-1}, w_{n-2}, x_{n-1} and x_n, compute

$$\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}) + \beta(1+\alpha)(w_{n-1} - x_n) \\ \qquad\quad -\alpha\beta(w_{n-2} - x_{n-1}), \\ y_n = \mathcal{J}_{\varpi_n \mathcal{A}}(w_n - \varpi_n(\mathcal{B} + \mathcal{C})w_n), \\ z_n = \mathcal{P}_{\mathcal{X}}(y_n + \varpi_n(\mathcal{B}w_n - \mathcal{B}y_n)). \end{cases}$$

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Algorithm 1 (Contd)

3: Update the stepsize as: If

$$\begin{cases} \|\mathcal{B}w_n - \mathcal{B}y_n\| > \dfrac{\mu_0}{\varpi_n}\|w_n - y_n\| \\ \text{then} \quad \varpi_{n+1} = \dfrac{\mu_1\|w_n - y_n\|}{\|\mathcal{B}w_n - \mathcal{B}y_n\|}, \\ \text{otherwise,} \quad \varpi_{n+1} = (1+\delta_n)\varpi_n. \end{cases}$$

4: Compute

$$x_{n+1} = (1-\theta_n)w_n + \theta_n z_n.$$

Continue again from **2** by setting $n := n+1$.

Some of Our Contributions

Algorithm 2

Proximal Forward-Reflected Scheme with Momentum

1: Given $\theta \geq 0$, select $x_1,\ x_0,\ z_1 \in \mathbb{R}^n$, and set $n = 1$.

2: Given $x_{n-1},\ x_n$ and z_n, compute

$$\begin{cases} y_n = \dfrac{1}{1+\theta}x_n + \dfrac{\theta}{1+\theta}z_n, \\ x_{n+1} = \mathrm{Prox}_{h+\iota_{\mathcal{K}}}\left(y_n - \dfrac{1}{\gamma}(2\mathcal{T}x_n - \mathcal{T}x_{n-1})\right), \\ z_{n+1} = \dfrac{1}{1+\theta}x_{n+1} + \dfrac{\theta}{1+\theta}z_n. \end{cases}$$

3: Set $n \leftarrow n+1$, and go to **2**.

[C.E. Nwakpa, et al (Optimization, 2025)]

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Scheme 3

Dynamical System for Solving Mixed Variational Inequality Problem

Given $\lambda > 0$ and $x(0) \in \mathcal{H}$, the trajectories generated by our dynamical system are given as follows

$$\begin{cases} y(t) = prox_{\lambda h}(x(t) - \lambda \mathcal{T} x(t)) \\ \dot{x}(t) + x(t) = y(t) + \lambda \left[\mathcal{T}(x(t)) - \mathcal{T}(y(t))\right] \\ x(0) = x_0. \end{cases} \tag{1}$$


Thank
You



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